



GANDHI SCHOOL OF ENGINEERING

Brahmapur, Odisha

E-Notes

Engineering Physics

Complete E-Notes

B.Tech 1st Year · Semester 1 · Circuit Branch

Academic Year 2026-27



How to use these notes

These notes follow the official 2026-27 syllabus for Physics (3-0-0, 3 credits, 45 contact hours, 100 university marks + 50 internal). Each module has these parts:

- **Theory:** definitions, derivations and physical explanations in exam language.
- **Formula boxes:** every formula you must remember, in shaded boxes.
- **Solved numericals:** step-by-step, with units at every step.
- **Practice problems with solutions.**
- **MCQs:** each with the correct option and a one-line reason.

At the end: 2-mark questions with answers, a model question paper with solutions, and a formula sheet.

Module	Topic	Hours
I	Oscillations: SHM (mechanical and electrical), damped oscillations, energy decay, quality factor, forced oscillations, resonance	9
II	Waves and Optics: wave equation, superposition, coherence, thin-film interference, Newton's rings, diffraction, single slit, grating	9
III	Electromagnetism: gradient, divergence, curl, Gauss and Stokes theorems, Maxwell's equations, EM waves in vacuum and conducting media	9
IV	Quantum Physics: wave-particle duality, phase and group velocity, de Broglie waves, wave function, operators, Schrödinger equation, particle in a box	9
V	Lasers: characteristics, Einstein coefficients, population inversion, three- and four-level schemes, Ruby and He-Ne lasers	8

Physical constants used in this book

Quantity	Symbol	Value
Speed of light in vacuum	c	$3 \times 10^8 \text{ m/s}$
Planck's constant	h	$6.626 \times 10^{-34} \text{ J s}$
Reduced Planck constant	$\hbar = h/2\pi$	$1.055 \times 10^{-34} \text{ J s}$
Electron charge	e	$1.602 \times 10^{-19} \text{ C}$
Electron mass	m_e	$9.11 \times 10^{-31} \text{ kg}$
Proton mass	m_p	$1.673 \times 10^{-27} \text{ kg}$



Quantity	Symbol	Value
Boltzmann constant	k_B	$1.38 \times 10^{-23} \text{ J/K}$
Permittivity of free space	ϵ_0	$8.854 \times 10^{-12} \text{ F/m}$
Permeability of free space	μ_0	$4\pi \times 10^{-7} \text{ H/m}$
1 electron-volt	1 eV	$1.602 \times 10^{-19} \text{ J}$
1 ångström	1 Å	10^{-10} m

Reference books (from syllabus)

- Ian G. Main, Oscillations and Waves in Physics, Cambridge University Press.
- H.J. Pain, The Physics of Vibrations and Waves, Wiley.
- E. Hecht, Optics, Pearson. A. Ghatak, Optics, McGraw Hill.
- O. Svelto, Principles of Lasers, Springer.



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Physics

Subject: Physics · Semester 1 (Circuit branch) · 3-0-0 · 3 credits · 45 hours · University 100 + Internal 50

Module I: Oscillations

An oscillation is any motion that repeats itself about an equilibrium position. A pendulum, a mass on a spring, the charge in an LC circuit, the atoms in a crystal and the electrons in an antenna all oscillate. The same differential equation describes all of them, so learning it once lets you analyse mechanical and electrical systems together.

1.1 Simple harmonic motion (SHM)

A particle performs **simple harmonic motion** if its acceleration is proportional to its displacement from a fixed point and always directed towards that point:

$$a = d^2x/dt^2 = -\omega^2 x \text{ (equation of SHM: } d^2x/dt^2 + \omega^2 x = 0)$$

The minus sign shows the restoring nature of the force. The general solution is

$$x(t) = A \sin(\omega t + \phi) \text{ or } x(t) = A \cos(\omega t + \phi)$$

- **A** = amplitude (maximum displacement), **ω** = angular frequency (rad/s), **ϕ** = initial phase (epoch), **$(\omega t + \phi)$** = phase.
- **Time period** $T = 2\pi/\omega$; **frequency** $\nu = 1/T = \omega/2\pi$.

$$\text{Velocity } v = dx/dt = A\omega \cos(\omega t + \phi) = \pm \omega \sqrt{A^2 - x^2}; v_{\max} = A\omega \text{ at } x = 0$$

$$\text{Acceleration } a = -\omega^2 x; a_{\max} = \omega^2 A \text{ at } x = \pm A$$

Energy in SHM. For a mass m on a spring of constant k ($\omega^2 = k/m$):

$$KE = (1/2)mv^2 = (1/2)m\omega^2(A^2 - x^2) \quad PE = (1/2)kx^2 = (1/2)m\omega^2x^2$$

$$\text{Total energy } E = KE + PE = (1/2)m\omega^2A^2 = (1/2)kA^2 \text{ (constant, } \propto A^2)$$

KE is maximum at the mean position and PE is maximum at the extremes. The time-averaged KE and PE are each $E/2$.

1.2 Mechanical simple harmonic oscillators

Spring-mass system. A mass m attached to a spring of stiffness k . When displaced by x , the restoring force is $F = -kx$ (Hooke's law). Newton's second law: $m d^2x/dt^2 = -kx \Rightarrow d^2x/dt^2 + (k/m)x = 0$.

$$\omega = \sqrt{k/m}, T = 2\pi\sqrt{m/k}$$

Simple pendulum. A bob of mass m on a light string of length l . For small angle θ , the restoring torque gives $d^2\theta/dt^2 + (g/l)\theta = 0$.

$$\omega = \sqrt{g/l}, T = 2\pi\sqrt{l/g}$$

Compound (bar) pendulum. A rigid body oscillating about a horizontal axis at distance l from its centre of gravity: $T = 2\pi\sqrt{(k^2 + l^2)/(gl)}$, where k is the radius of gyration about the CG. (This is used in the Physics Lab bar-pendulum experiment.)

Torsional pendulum. A disc suspended by a wire of torsional constant C : $T = 2\pi\sqrt{I/C}$.

Springs in combination. Series: $1/k = 1/k_1 + 1/k_2$. Parallel: $k = k_1 + k_2$.

1.3 Electrical simple harmonic oscillator (LC circuit)

A charged capacitor C connected across an inductor L (with no resistance) exchanges energy between the electric field of the capacitor and the magnetic field of the inductor.

By Kirchhoff's voltage law: $L \frac{dI}{dt} + q/C = 0$. With $I = dq/dt$:

$$d^2q/dt^2 + (1/LC) q = 0 \Rightarrow q = q_0 \cos(\omega_0 t + \phi), \omega_0 = 1/\sqrt{LC}, \nu_0 = 1/(2\pi\sqrt{LC})$$

Mechanical-electrical analogy (very important):

Mechanical quantity	Electrical analogue
Displacement x	Charge q
Velocity $v = dx/dt$	Current $I = dq/dt$
Mass m (inertia)	Inductance L
Spring constant k	$1/C$ (inverse capacitance)
Damping coefficient b (or r)	Resistance R
Driving force $F_0 \cos \omega t$	Driving emf $E_0 \cos \omega t$
$KE = (1/2)mv^2$	Magnetic energy $= (1/2)LI^2$
$PE = (1/2)kx^2$	Electric energy $= q^2/2C$

Total energy in the LC circuit: $E = q_0^2/(2C) = (1/2)LI_0^2$, constant.

1.4 Damped harmonic oscillator

In a real system friction or resistance removes energy, so the amplitude decreases. For small velocities the damping force is proportional to velocity: $F_d = -b v$ (b = damping coefficient, unit kg/s).

Equation of motion.

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \Rightarrow \frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0, \text{ where } 2\gamma = b/m, \omega_0^2 = k/m$$

Try $x = e^{\alpha t}$: $\alpha^2 + 2\gamma\alpha + \omega_0^2 = 0 \Rightarrow \alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$. Three cases arise.

(a) Heavy damping (over-damping), $\gamma > \omega_0$. Both roots are real and negative:

$$x = e^{-\gamma t} [C_1 e^{\beta t} + C_2 e^{-\beta t}], \beta = \sqrt{\gamma^2 - \omega_0^2}$$

The body returns slowly to equilibrium without oscillating (e.g. a door closer, a pendulum in thick oil).

(b) Critical damping, $\gamma = \omega_0$. Repeated root $\alpha = -\gamma$:

$$x = (C_1 + C_2 t) e^{-\gamma t}$$

The body returns to equilibrium in the shortest possible time without oscillating. Used in moving-coil galvanometers, voltmeters and car shock absorbers.

(c) Light damping (under-damping), $\gamma < \omega_0$. Complex roots:

$$x = A_0 e^{-\gamma t} \cos(\omega' t + \phi), \omega' = \sqrt{\omega_0^2 - \gamma^2}$$

The body oscillates with an exponentially decaying amplitude $A_0 e^{-\gamma t}$ and a slightly smaller frequency ω' than the natural frequency. Examples: a pendulum in air, a tuning fork, an LCR circuit with small R .

Logarithmic decrement. The ratio of successive amplitudes (one period T' apart) is constant: $A_n/A_{n+1} = e^{\gamma T'}$.

$$\lambda = \ln(A_n/A_{n+1}) = \gamma T'$$

Electrical damped oscillator (series LCR, no source). $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + q/C = 0$
 $\Rightarrow \gamma = R/2L$, $\omega_0 = 1/\sqrt{LC}$. Oscillatory discharge if $R < 2\sqrt{L/C}$; critical if $R = 2\sqrt{L/C}$; non-oscillatory if $R > 2\sqrt{L/C}$.

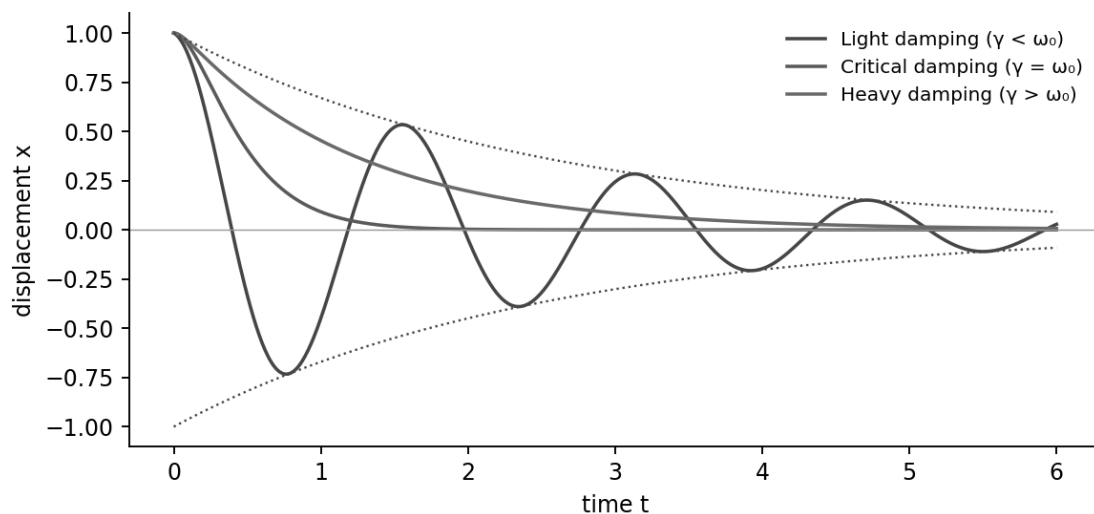


Figure 1.1: Displacement-time graphs for light, critical and heavy damping. Dotted lines show the decaying envelope $\pm A_0 e^{-\gamma t}$.

1.5 Energy decay in a damped harmonic oscillator

For light damping the energy is proportional to the square of the amplitude:

$$E(t) = (1/2)kA_0^2 e^{-2\gamma t} = E_0 e^{-2\gamma t} = E_0 e^{-t/\tau}$$

$$\text{Relaxation time (energy decay time)} \tau = 1/(2\gamma) = m/b$$

τ is the time in which the energy falls to $1/e$ (about 37%) of its initial value. The amplitude falls to $1/e$ of its initial value in time $1/\gamma = 2\tau$.

Rate of energy loss. $dE/dt = -bv^2$, i.e. the power dissipated against the damping force. Averaged over a cycle, $\langle dE/dt \rangle = -E/\tau$.

1.6 Quality factor (Q-factor)

The quality factor measures how good an oscillator is at storing energy compared with how fast it loses energy.

$$Q = 2\pi \times (\text{energy stored})/(\text{energy lost per cycle}) = \omega \tau = \omega/(2\gamma)$$

$$\text{Mechanical: } Q = m\omega_0/b = \sqrt{mk}/b \quad \text{Electrical (series LCR): } Q = \omega_0 L/R = (1/R)\sqrt{L/C}$$

- A high Q means low damping and many oscillations before the motion dies out (tuning fork $Q \approx 1000$, quartz crystal $Q \approx 10^5$ - 10^6).
- A low Q means heavy damping (car suspension $Q \approx 1$).

- Q also measures the sharpness of resonance in forced oscillations: $Q = \omega_0/\Delta\omega$, where $\Delta\omega$ is the bandwidth (full width at half-power).
- The number of oscillations for the energy to fall to $1/e$ is $Q/2\pi$.

1.7 Forced mechanical oscillator

When an external periodic force $F(t) = F_0 \cos \omega t$ acts on a damped oscillator, the system is forced to oscillate at the driving frequency ω .

$$m \, d^2x/dt^2 + b \, dx/dt + kx = F_0 \cos \omega t \Rightarrow d^2x/dt^2 + 2\gamma \, dx/dt + \omega_0^2 x = f_0 \cos \omega t, f_0 = F_0/m$$

The complete solution is the sum of two parts:

- **Transient solution:** the solution of the damped equation without the force. It dies out as $e^{-\gamma t}$.
- **Steady-state solution:** a particular solution that oscillates at the driving frequency and persists as long as the force acts.

1.8 Steady-state motion of a forced damped oscillator

Try $x = A \cos(\omega t - \delta)$. Substituting and comparing coefficients of $\cos \omega t$ and $\sin \omega t$:

$$A(\omega_0^2 - \omega^2) = f_0 \cos \delta, 2\gamma\omega A = f_0 \sin \delta$$

Squaring and adding, and dividing:

$$\text{Amplitude } A(\omega) = f_0 / \sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} = F_0 / \sqrt{[m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2]}$$

$$\text{Phase lag } \tan \delta = 2\gamma\omega/(\omega_0^2 - \omega^2)$$

Behaviour at different driving frequencies.

Driving frequency	Amplitude	Phase lag δ of displacement behind force
$\omega \ll \omega_0$ (low)	$A \approx F_0/k$ (controlled by stiffness)	$\delta \approx 0$ (in phase)
$\omega = \omega_0$ (resonance)	$A = F_0/(b\omega_0)$ (controlled by damping)	$\delta = \pi/2$
$\omega \gg \omega_0$ (high)	$A \approx F_0/(m\omega^2)$ (controlled by mass)	$\delta \approx \pi$ (opposite phase)

1.9 Amplitude resonance

The amplitude is maximum when the denominator $(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2$ is minimum. Differentiating with respect to ω and equating to zero:

$$\text{Amplitude resonance frequency } \omega_r = \sqrt{(\omega_0^2 - 2\gamma^2)}$$

$$\text{Maximum amplitude } A_{\max} = f_0/(2\gamma\sqrt{(\omega_0^2 - \gamma^2)})$$

- ω_r is slightly less than ω_0 . For small damping $\omega_r \approx \omega_0$.
- The smaller the damping, the taller and sharper the resonance peak. With zero damping, $A \rightarrow \infty$ at $\omega = \omega_0$.

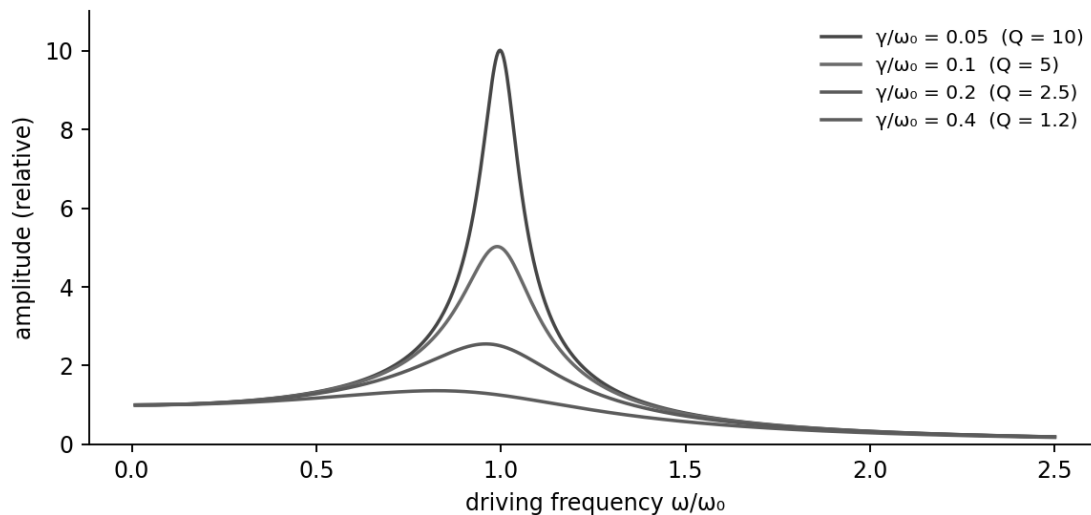


Figure 1.2: Amplitude resonance curves for different damping. Smaller damping (higher Q) gives a taller, sharper peak.

1.10 Velocity resonance and power absorption

Velocity amplitude $v_0 = \omega A = f_0 \omega / \sqrt{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2]}$. This is maximum exactly at $\omega = \omega_0$ (**velocity resonance**), where the velocity is in phase with the force.

The average power absorbed from the driving force is

$$\langle P \rangle = (F_0^2 / 2m) \times 2\gamma \omega^2 / [(\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2], \text{ maximum } \langle P \rangle_{\text{max}} = F_0^2 / (2b) \text{ at } \omega = \omega_0$$

Bandwidth and sharpness of resonance. The frequencies ω_1, ω_2 at which the power falls to half its maximum are the **half-power points**. For small damping $\omega_2 - \omega_1 = 2\gamma$. Hence

$$Q = \omega_0 / (\omega_2 - \omega_1) = \omega_0 / (2\gamma)$$

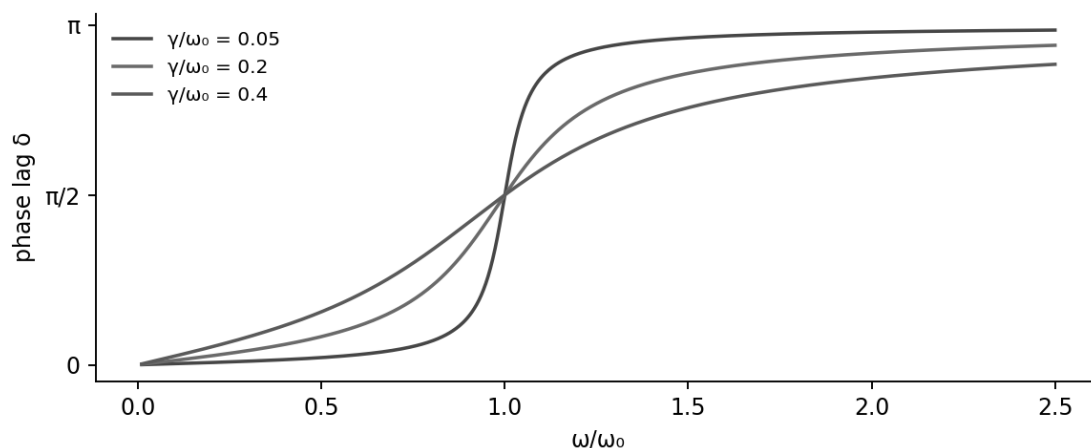


Figure 1.3: Phase lag of displacement behind the driving force: 0 at low frequency, $\pi/2$ at resonance, π at high frequency.

1.11 Forced electrical oscillator (series LCR with AC source)

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + q/C = E_0 \cos \omega t$$

This is the same equation as the mechanical one with $m \rightarrow L$, $b \rightarrow R$, $k \rightarrow 1/C$, $F_0 \rightarrow E_0$. The steady-state current is

$$I_0 = E_0/Z, Z = \sqrt{R^2 + (\omega L - 1/(\omega C))^2}, \tan \phi = (\omega L - 1/(\omega C))/R$$

- **Series resonance** at $\omega_0 = 1/\sqrt{LC}$: $X_L = X_C$, $Z = R$ (minimum), current is maximum $I_0 = E_0/R$ and in phase with the voltage.
- $Q = \omega_0 L/R = 1/(\omega_0 CR)$. The voltage across L (or C) at resonance is Q times the source voltage (voltage magnification).
- Tuning a radio selects one station by adjusting C so that ω_0 matches the station's frequency. A high Q gives better selectivity.

1.12 Examples of resonance in engineering

- **Useful:** radio and TV tuning circuits, microwave ovens (water molecules absorb at 2.45 GHz), MRI, musical instruments, quartz clocks.
- **Harmful:** the Tacoma Narrows bridge collapse (1940, wind-induced oscillations), soldiers breaking step on bridges, machine vibrations at critical speeds, earthquakes matching building frequencies. Engineers add damping (shock absorbers, tuned mass dampers) to avoid large resonant amplitudes.

1.13 Common mistakes to avoid in Module I

- Mixing up ω (rad/s) and ν (Hz): $\omega = 2\pi\nu$.
- Writing $\omega' = \omega_0$ for a damped oscillator. It is $\sqrt{(\omega_0^2 - \gamma^2)}$.
- Forgetting the factor 2 between amplitude decay ($e^{-\gamma t}$) and energy decay ($e^{-2\gamma t}$).
- Confusing the resonance conditions: amplitude resonance at $\sqrt{(\omega_0^2 - 2\gamma^2)}$, velocity and power resonance at ω_0 .
- Using the wrong damping symbol: some books write the equation as $\ddot{x} + 2b\dot{x} + \omega_0^2 x = 0$ with $b = r/2m$. Always state your notation.

1.14 Long-answer derivations (exam style)

(A) Energy of a simple harmonic oscillator is constant. Let $x = A \sin(\omega t + \phi)$. Then $v = A\omega \cos(\omega t + \phi)$.

$KE = (1/2)mv^2 = (1/2)mA^2\omega^2 \cos^2(\omega t + \phi)$. $PE = (1/2)kx^2 = (1/2)m\omega^2 A^2 \sin^2(\omega t + \phi)$, using $k = m\omega^2$.

Total $E = (1/2)m\omega^2 A^2 [\cos^2 + \sin^2] = (1/2)m\omega^2 A^2$, which is independent of time and of position. The average of $\cos^2$ or $\sin^2$ over one period is $1/2$, so $\langle KE \rangle = \langle PE \rangle = E/2$.

(B) Solution of the damped oscillator equation. Equation: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$. Try $x = e^{\alpha t}$: $\alpha^2 + 2\gamma\alpha + \omega_0^2 = 0$, $\alpha = -\gamma \pm \sqrt{(\gamma^2 - \omega_0^2)}$.

- Heavy damping ($\gamma > \omega_0$): $\beta = \sqrt{(\gamma^2 - \omega_0^2)}$ is real and less than γ , so both roots $-\gamma \pm \beta$ are negative. $x = e^{-\gamma t}(C_1 e^{\beta t} + C_2 e^{-\beta t})$ decays without oscillation.
- Critical damping ($\gamma = \omega_0$): double root $-\gamma$. The second independent solution is $te^{-\gamma t}$, so $x = (C_1 + C_2 t)e^{-\gamma t}$. For $x(0) = A$, $\dot{x}(0) = 0$: $C_1 = A$, $C_2 = \gamma A$, and $x = A(1 + \gamma t)e^{-\gamma t}$.
- Light damping ($\gamma < \omega_0$): $\sqrt{(\gamma^2 - \omega_0^2)} = i\omega'$ with $\omega' = \sqrt{(\omega_0^2 - \gamma^2)}$. $x = e^{-\gamma t}(C_1 e^{i\omega' t} + C_2 e^{-i\omega' t})$, which for real x is $x = A_0 e^{-\gamma t} \cos(\omega' t + \phi)$.

Why critical damping is fastest. In heavy damping, the slower exponent $-\gamma + \beta$ is closer to zero than $-\gamma$, so the decay is slower. In light damping the system overshoots and oscillates. Critical damping has the fastest pure decay without overshoot.

(C) Energy decay and Q for light damping. For $\gamma \ll \omega_0$, the energy averaged over a cycle is $E = (1/2)m\omega_0^2 A^2$, with $A = A_0 e^{-\gamma t}$. So $E = E_0 e^{-2\gamma t}$. Power lost: $-dE/dt = 2\gamma E$. Energy lost per cycle (period $T = 2\pi/\omega_0$) = $2\gamma E \times 2\pi/\omega_0$. Hence

$$Q = 2\pi E / (\text{energy lost per cycle}) = 2\pi E / (2\gamma E \times 2\pi/\omega_0) = \omega_0 / (2\gamma).$$

(D) Steady-state amplitude of a forced oscillator using complex numbers. Write the force as $F_0 e^{i\omega t}$ and try $x = X e^{i\omega t}$ (X complex). Substituting in $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f_0 e^{i\omega t}$: $(-\omega^2 + 2i\gamma\omega + \omega_0^2)X = f_0$. So $X = f_0 / [(\omega_0^2 - \omega^2) + 2i\gamma\omega]$. The magnitude $|X| = f_0 / \sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$ is the amplitude, and the argument gives the phase lag $\tan \delta = 2\gamma\omega / (\omega_0^2 - \omega^2)$.

(E) Condition for amplitude resonance. Let $D(\omega) = (\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2$. $dD/d\omega = 2(\omega_0^2 - \omega^2)(-2\omega) + 8\gamma^2\omega = 0 \Rightarrow \omega^2 = \omega_0^2 - 2\gamma^2$. This is a minimum of D , hence a maximum of A , provided $\omega_0^2 > 2\gamma^2$. If the damping is so large that $2\gamma^2 \geq \omega_0^2$, there is no resonance peak and the amplitude decreases steadily with frequency.

(F) Power absorbed and half-power bandwidth. Average power = $\langle F v \rangle = (1/2)F_0 v_0 \cos(\text{phase between } F \text{ and } v)$. This gives $\langle P \rangle = (F_0^2/2m) \cdot 2\gamma\omega^2 / [(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]$. Near resonance put $\omega \approx \omega_0$, so $\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)$. Then $\langle P \rangle \approx (F_0^2/4m\gamma) \times \gamma^2 / [(\omega_0 - \omega)^2 + \gamma^2]$. This is half its maximum when $(\omega_0 - \omega)^2 = \gamma^2$, i.e. $\omega = \omega_0 \pm \gamma$. So the full width at half power is $\Delta\omega = 2\gamma$, and $Q = \omega_0/\Delta\omega$.

(G) Series LCR circuit as a forced oscillator. KVL: $L di/dt + Ri + q/C = E_0 \cos \omega t$. In phasor form, $Z = R + i(\omega L - 1/\omega C)$, $|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$. The current amplitude $I_0 = E_0/|Z|$ is maximum when $\omega L = 1/\omega C$, i.e. $\omega_0 = 1/\sqrt{LC}$. At resonance the voltages across L and C are equal and opposite, each equal to QE_0 , and cancel each other.

1.15 Comparison tables for quick revision

Property	Undamped SHM	Damped (light)	Forced (steady state)
Equation	$\ddot{x} + \omega_0^2 x = 0$	$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$	$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f_0 \cos \omega t$
Frequency	ω_0	$\omega' = \sqrt{(\omega_0^2 - \gamma^2)}$	ω (driving)
Amplitude	Constant A	$A_0 e^{-\gamma t}$ (decays)	Constant, depends on ω
Energy	Constant	$E_0 e^{-2\gamma t}$	Constant (supplied by driver)
Example	Ideal pendulum	Pendulum in air	Radio tuning circuit

The mechanical-electrical correspondence for damped and forced oscillators:

Mechanical system	Electrical system
$m \ddot{x} + b \dot{x} + kx = F_0 \cos \omega t$	$L \ddot{q} + R \dot{q} + q/C = E_0 \cos \omega t$
$\omega_0 = \sqrt{k/m}$	$\omega_0 = 1/\sqrt{LC}$
$\gamma = b/2m$	$\gamma = R/2L$
$Q = \sqrt{mk}/b$	$Q = (1/R)\sqrt{L/C}$
Critical when $b = 2\sqrt{mk}$	Critical when $R = 2\sqrt{L/C}$

1.16 Solved numericals: Module I

Problem 1. A particle executes SHM with amplitude 5 cm and period 4 s. Find its maximum velocity, maximum acceleration, and velocity when the displacement is 3 cm.

$$\omega = 2\pi/T = 2\pi/4 = \pi/2 \text{ rad/s} \approx 1.571 \text{ rad/s. } A = 0.05 \text{ m.}$$

$$v_{\max} = A\omega = 0.05 \times 1.571 = 0.0785 \text{ m/s.}$$

$$a_{\max} = \omega^2 A = (2.467)(0.05) = 0.123 \text{ m/s}^2.$$

$$v \text{ at } x = 3 \text{ cm: } v = \omega\sqrt{A^2 - x^2} = 1.571 \times \sqrt{(0.0025 - 0.0009)} = 1.571 \times 0.04 = 0.0628 \text{ m/s.}$$

Answer: $v_{\max} = 7.85 \text{ cm/s}$; $a_{\max} = 12.3 \text{ cm/s}^2$; $v = 6.28 \text{ cm/s}$ at $x = 3 \text{ cm}$.

Problem 2. A mass of 0.5 kg attached to a spring oscillates with a period of 0.4 s. Find the spring constant and the total energy if the amplitude is 4 cm.

$$k = 4\pi^2 m/T^2 = 4 \times 9.87 \times 0.5/0.16 = 123.4 \text{ N/m.}$$

$$E = (1/2)kA^2 = 0.5 \times 123.4 \times (0.04)^2 = 0.0987 \text{ J.}$$

Answer: $k \approx 123 \text{ N/m}$; $E \approx 0.099 \text{ J}$.

Problem 3. At what displacement is the kinetic energy of a particle in SHM equal to its potential energy?

$$(1/2)m\omega^2(A^2 - x^2) = (1/2)m\omega^2 x^2 \Rightarrow A^2 - x^2 = x^2 \Rightarrow x = A/\sqrt{2}.$$

Answer: $x = \pm A/\sqrt{2} \approx \pm 0.707A$.

Problem 4. Find the length of a seconds pendulum ($T = 2 \text{ s}$) at a place where $g = 9.8 \text{ m/s}^2$.

$$T = 2\pi\sqrt{l/g} \Rightarrow l = gT^2/(4\pi^2) = 9.8 \times 4/39.48.$$

Answer: $l \approx 0.993 \text{ m}$.

Problem 5. An LC circuit has $L = 10 \text{ mH}$ and $C = 1 \text{ }\mu\text{F}$. Find its natural frequency of oscillation.

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{(10^{-2} \times 10^{-6})} = 1/\sqrt{(10^{-8})} = 10^4 \text{ rad/s.}$$

$$\nu_0 = \omega_0/2\pi = 10^4/6.283 = 1591.5 \text{ Hz.}$$

Answer: $\omega_0 = 10^4 \text{ rad/s}$; $\nu_0 \approx 1.59 \text{ kHz}$.

Problem 6. In an LC circuit, the maximum charge on the capacitor ($C = 4 \text{ }\mu\text{F}$) is $20 \text{ }\mu\text{C}$ and $L = 25 \text{ mH}$. Find the maximum current and the total energy.

$$\omega_0 = 1/\sqrt{25 \times 10^{-3} \times 4 \times 10^{-6}} = 1/\sqrt{(10^{-7})} = 3162 \text{ rad/s.}$$

$$I_{\max} = \omega_0 q_0 = 3162 \times 20 \times 10^{-6} = 0.0632 \text{ A.}$$

$$E = q_0^2/2C = (20 \times 10^{-6})^2/(2 \times 4 \times 10^{-6}) = 4 \times 10^{-10}/(8 \times 10^{-6}) = 5 \times 10^{-5} \text{ J.}$$

Answer: $I_{\max} \approx 63.2 \text{ mA}$; $E = 50 \text{ }\mu\text{J}$.

Problem 7. A damped oscillator has $m = 0.2 \text{ kg}$, $k = 80 \text{ N/m}$ and $b = 0.4 \text{ kg/s}$. Is it lightly damped? Find ω_0 , γ , ω' , relaxation time τ and Q .

$$\omega_0 = \sqrt{k/m} = \sqrt{400} = 20 \text{ rad/s. } \gamma = b/2m = 0.4/0.4 = 1 \text{ s}^{-1}.$$

Since $\gamma = 1 < \omega_0 = 20$, the motion is lightly damped (oscillatory).

$$\omega' = \sqrt{(\omega_0^2 - \gamma^2)} = \sqrt{(400 - 1)} = 19.97 \text{ rad/s.}$$

$$\tau = 1/(2\gamma) = 0.5 \text{ s. } Q = \omega_0/(2\gamma) = 20/2 = 10.$$

Answer: Lightly damped; $\omega_0 = 20 \text{ rad/s}$, $\gamma = 1 \text{ s}^{-1}$, $\omega' \approx 19.97 \text{ rad/s}$, $\tau = 0.5 \text{ s}$, $Q = 10$.

Problem 8. The amplitude of a damped oscillator falls from 10 cm to 5 cm in 20 oscillations of period 2 s each. Find the damping constant γ and the logarithmic decrement.

$$A = A_0 e^{-\gamma t}: 5 = 10 e^{-\gamma(40)} \Rightarrow \gamma = \ln 2/40 = 0.693/40 = 0.0173 \text{ s}^{-1}.$$

$$\lambda = \gamma T = 0.0173 \times 2 = 0.0347.$$

Answer: $\gamma \approx 0.0173 \text{ s}^{-1}$; $\lambda \approx 0.0347$.

Problem 9. A tuning fork of frequency 256 Hz has $Q = 1000$. Find the relaxation time and the time for its energy to fall to $1/e$ of its initial value.

$$\omega = 2\pi \times 256 = 1608.5 \text{ rad/s. } Q = \omega\tau \Rightarrow \tau = Q/\omega = 1000/1608.5 = 0.622 \text{ s.}$$

The energy falls to $1/e$ of its initial value in time τ .

Answer: $\tau \approx 0.62 \text{ s}$ (energy falls to $1/e$ in 0.62 s).

Problem 10. How many oscillations does an oscillator with $Q = 2000$ and frequency 500 Hz make before its energy falls to $1/e$?

$$\text{Number of oscillations in time } \tau = \nu \tau = \nu Q/\omega = Q/2\pi = 2000/6.283.$$

Answer: About 318 oscillations.

Problem 11. A series LCR circuit has $L = 1 \text{ mH}$, $C = 0.1 \text{ }\mu\text{F}$. Find the critical resistance and state the nature of discharge if $R = 50 \text{ }\Omega$.

$$R_c = 2\sqrt{L/C} = 2\sqrt{(10^{-3}/10^{-7})} = 2\sqrt{(10^4)} = 200 \text{ }\Omega.$$

$R = 50 \text{ }\Omega < 200 \text{ }\Omega$, so the discharge is oscillatory (lightly damped).

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{(10^{-10})} = 10^5 \text{ rad/s; } \gamma = R/2L = 50/0.002 = 2.5 \times 10^4 \text{ s}^{-1}; \omega' = \sqrt{(10^{10} - 6.25 \times 10^8)} = 9.68 \times 10^4 \text{ rad/s.}$$

Answer: $R_c = 200 \text{ }\Omega$; oscillatory discharge with $\omega' \approx 9.68 \times 10^4 \text{ rad/s}$.

Problem 12. A series LCR circuit with $L = 0.5 \text{ H}$, $C = 8 \text{ }\mu\text{F}$ and $R = 10 \text{ }\Omega$ is connected to a 100 V AC source of variable frequency. Find the resonance frequency, the current at resonance, the Q-factor and the voltage across the inductor at resonance.

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{(0.5 \times 8 \times 10^{-6})} = 1/\sqrt{(4 \times 10^{-6})} = 500 \text{ rad/s; } \nu_0 = 500/2\pi = 79.6 \text{ Hz.}$$

$$\text{At resonance } Z = R \Rightarrow I = 100/10 = 10 \text{ A.}$$

$$Q = \omega_0 L/R = 500 \times 0.5/10 = 25.$$

$$V_L = I \omega_0 L = 10 \times 250 = 2500 \text{ V } (= Q \times 100 \text{ V}).$$

Answer: $\nu_0 \approx 79.6 \text{ Hz}$; $I = 10 \text{ A}$; $Q = 25$; $V_L = 2500 \text{ V}$.

Problem 13. A body of mass 1 kg with $\omega_0 = 10 \text{ rad/s}$ and $\gamma = 0.5 \text{ s}^{-1}$ is driven by $F = 5 \cos \omega t \text{ N}$. Find the amplitude at $\omega = 5 \text{ rad/s}$, $\omega = 10 \text{ rad/s}$, and the amplitude-resonance frequency.

$$f_0 = F_0/m = 5 \text{ m/s}^2.$$

$$\omega = 5: A = 5/\sqrt{[(100 - 25)^2 + 4(0.25)(25)]} = 5/\sqrt{(5625 + 25)} = 5/75.17 = 0.0665 \text{ m}.$$

$$\omega = 10: A = 5/\sqrt{[0 + 4(0.25)(100)]} = 5/10 = 0.5 \text{ m}.$$

$$\omega_r = \sqrt{(\omega_0^2 - 2\gamma^2)} = \sqrt{(100 - 0.5)} = 9.975 \text{ rad/s}.$$

Answer: 6.65 cm at 5 rad/s; 50 cm at 10 rad/s; $\omega_r \approx 9.97 \text{ rad/s}$.

Problem 14. The resonance frequency of an oscillator is 1000 Hz and its half-power bandwidth is 20 Hz. Find the Q-factor and the damping constant γ .

$$Q = \nu_0/\Delta\nu = 1000/20 = 50.$$

$$\Delta\omega = 2\gamma \Rightarrow \gamma = \pi \Delta\nu = \pi \times 20 = 62.8 \text{ s}^{-1}.$$

Answer: $Q = 50$; $\gamma \approx 62.8 \text{ s}^{-1}$.

Problem 15. A spring of constant 200 N/m is cut into two equal halves and the halves are joined in parallel. Find the new spring constant and the period with a 2 kg mass.

Each half has $k' = 2 \times 200 = 400 \text{ N/m}$ (a shorter spring is stiffer). In parallel: $k = 800 \text{ N/m}$.

$$T = 2\pi\sqrt{(2/800)} = 2\pi \times 0.05 = 0.314 \text{ s}.$$

Answer: $k = 800 \text{ N/m}$; $T \approx 0.314 \text{ s}$.

Problem 16. The displacement of an oscillator is $x = 0.02 e^{-0.1t} \cos(4\pi t) \text{ m}$. Find the natural frequency, damping constant, quality factor and the time for the amplitude to halve.

$$\gamma = 0.1 \text{ s}^{-1}, \omega' = 4\pi = 12.57 \text{ rad/s}. \omega_0 = \sqrt{(\omega'^2 + \gamma^2)} \approx 12.57 \text{ rad/s}.$$

$$Q = \omega_0/2\gamma = 12.57/0.2 = 62.8.$$

$$\text{Half amplitude: } e^{-0.1t} = 1/2 \Rightarrow t = \ln 2/0.1 = 6.93 \text{ s}.$$

Answer: $\omega_0 \approx 12.57 \text{ rad/s}$ (2 Hz); $\gamma = 0.1 \text{ s}^{-1}$; $Q \approx 63$; $t_{1/2} \approx 6.93 \text{ s}$.

1.17 More solved numericals: Module I

Problem 1. A particle of mass 0.1 kg executes SHM with amplitude 0.1 m. At its mean position its kinetic energy is $8 \times 10^{-3} \text{ J}$. Write the equation of motion if the initial phase is 45° .

$$KE_{\max} = (1/2)m\omega^2 A^2 \Rightarrow 8 \times 10^{-3} = 0.5 \times 0.1 \times \omega^2 \times 0.01 \Rightarrow \omega^2 = 16 \Rightarrow \omega = 4 \text{ rad/s}.$$

Answer: $x = 0.1 \sin(4t + \pi/4) \text{ m}$.

Problem 2. A body in SHM has displacement $x = 0.05 \cos(\pi t/3) \text{ m}$. Find the time taken to go from $x = 0.05 \text{ m}$ to $x = 0.025 \text{ m}$.

$$0.025 = 0.05 \cos(\pi t/3) \Rightarrow \cos(\pi t/3) = 1/2 \Rightarrow \pi t/3 = \pi/3 \Rightarrow t = 1 \text{ s}.$$



Answer: 1 s.

Problem 3. A spring stretches 5 cm when a 200 g mass hangs from it. Find the period of vertical oscillations.

$$k = mg/x = 0.2 \times 9.8/0.05 = 39.2 \text{ N/m. } T = 2\pi\sqrt{(m/k)} = 2\pi\sqrt{(0.2/39.2)} = 2\pi \times 0.0714 = 0.449 \text{ s.}$$

$$(\text{Equivalently } T = 2\pi\sqrt{(x/g)} = 2\pi\sqrt{(0.05/9.8).})$$

Answer: $T \approx 0.45 \text{ s.}$

Problem 4. Two springs of constants 100 N/m and 300 N/m are joined in series and carry a mass of 3 kg. Find the frequency.

$$k = k_1 k_2 / (k_1 + k_2) = 30000/400 = 75 \text{ N/m. } \omega = \sqrt{(75/3)} = 5 \text{ rad/s.}$$

Answer: $\nu = 5/2\pi \approx 0.80 \text{ Hz.}$

Problem 5. A capacitor of 10 μF charged to 50 V is connected across an inductor of 0.1 H. Find the frequency, the maximum current and the energy stored.

$$\omega_0 = 1/\sqrt{(0.1 \times 10^{-5})} = 1/\sqrt{(10^{-6})} = 1000 \text{ rad/s; } \nu = 159 \text{ Hz.}$$

$$q_0 = CV = 5 \times 10^{-4} \text{ C; } I_0 = \omega_0 q_0 = 0.5 \text{ A.}$$

$$E = (1/2)CV^2 = 0.5 \times 10^{-5} \times 2500 = 0.0125 \text{ J (check: } (1/2)LI_0^2 = 0.5 \times 0.1 \times 0.25 = 0.0125 \text{ J).}$$

Answer: $\nu \approx 159 \text{ Hz; } I_0 = 0.5 \text{ A; } E = 12.5 \text{ mJ.}$

Problem 6. A pendulum oscillating in air has its amplitude reduced to half in 50 s. Find the time in which its energy falls to half.

$$A = A_0 e^{-\gamma t}; e^{-50\gamma} = 1/2 \Rightarrow \gamma = \ln 2/50 = 0.01386 \text{ s}^{-1}.$$

$$E = E_0 e^{-2\gamma t} = E_0/2 \Rightarrow t = \ln 2/(2\gamma) = 25 \text{ s.}$$

Answer: 25 s (half the time for amplitude).

Problem 7. For a damped oscillator the amplitude after 100 oscillations is 1/e of the initial value. If the period is 0.5 s, find γ , τ and Q.

$$100 \text{ oscillations take } 50 \text{ s} \Rightarrow \gamma \times 50 = 1 \Rightarrow \gamma = 0.02 \text{ s}^{-1}.$$

$$\tau = 1/(2\gamma) = 25 \text{ s. } \omega \approx 2\pi/0.5 = 12.57 \text{ rad/s} \Rightarrow Q = \omega\tau = 314.$$

Answer: $\gamma = 0.02 \text{ s}^{-1}; \tau = 25 \text{ s; } Q \approx 314.$

Problem 8. A mass of 0.5 kg on a spring ($k = 50 \text{ N/m}$) experiences a damping force $-bv$. Find b for critical damping.

$$\text{Critical: } b = 2\sqrt{(mk)} = 2\sqrt{(25)} = 10 \text{ kg/s.}$$

Answer: $b = 10 \text{ kg/s.}$

Problem 9. A series LCR circuit ($R = 20 \Omega$, $L = 0.2 \text{ H}$, $C = 50 \mu\text{F}$) is driven by a 220 V (rms), variable-frequency source. Find the resonant frequency, the rms current at resonance, Q and the bandwidth.

$$\omega_0 = 1/\sqrt{0.2 \times 5 \times 10^{-5}} = 1/\sqrt{10^{-5}} = 316.2 \text{ rad/s}; \nu_0 = 50.3 \text{ Hz.}$$

$$I_{\text{rms}} = 220/20 = 11 \text{ A. } Q = \omega_0 L/R = 316.2 \times 0.2/20 = 3.16.$$

$$\Delta\omega = R/L = 100 \text{ rad/s } (\Delta\nu = 15.9 \text{ Hz}).$$

Answer: $\nu_0 \approx 50.3 \text{ Hz}$; $I = 11 \text{ A}$; $Q \approx 3.16$; $\Delta\omega = 100 \text{ rad/s}$.

Problem 10. The same LCR circuit is driven at 40 Hz . Find the impedance and current.

$$\omega = 251.3 \text{ rad/s. } X_L = \omega L = 50.3 \Omega. X_C = 1/(\omega C) = 1/(251.3 \times 5 \times 10^{-5}) = 79.6 \Omega.$$

$$Z = \sqrt{20^2 + (50.3 - 79.6)^2} = \sqrt{400 + 858} = 35.5 \Omega. I = 220/35.5 = 6.2 \text{ A.}$$

Answer: $Z \approx 35.5 \Omega$ (capacitive); $I \approx 6.2 \text{ A}$.

Problem 11. A forced oscillator ($m = 2 \text{ kg}$, $k = 200 \text{ N/m}$, $b = 4 \text{ kg/s}$) is driven by $F = 10 \cos \omega t \text{ N}$. Find the amplitude at resonance ($\omega = \omega_0$) and the maximum average power absorbed.

$$\omega_0 = \sqrt{200/2} = 10 \text{ rad/s. At } \omega = \omega_0: A = F_0/(b\omega_0) = 10/(4 \times 10) = 0.25 \text{ m.}$$

$$P_{\text{max}} = F_0^2/(2b) = 100/8 = 12.5 \text{ W.}$$

Answer: $A = 25 \text{ cm}$; $P_{\text{max}} = 12.5 \text{ W}$.

Problem 12. The quality factor of a musical string is 500 and its frequency is 440 Hz . How long does it take for the energy to fall to $1/e$?

$$\tau = Q/\omega = 500/(2\pi \times 440) = 500/2764.6.$$

Answer: $\tau \approx 0.18 \text{ s}$.

1.18 Practice problems with solutions: Module I

Q1. The equation of an SHM is $x = 0.1 \sin(10\pi t + \pi/6) \text{ m}$. Find amplitude, frequency, period and initial phase.

$$\text{Compare with } x = A \sin(\omega t + \phi): A = 0.1 \text{ m, } \omega = 10\pi \text{ rad/s} \Rightarrow \nu = 5 \text{ Hz, } T = 0.2 \text{ s, } \phi = \pi/6.$$

Answer: $A = 0.1 \text{ m}$, $\nu = 5 \text{ Hz}$, $T = 0.2 \text{ s}$, $\phi = 30^\circ$.

Q2. A particle in SHM has velocities 4 cm/s and 3 cm/s at displacements 3 cm and 4 cm . Find amplitude and period.

$$v^2 = \omega^2(A^2 - x^2): 16 = \omega^2(A^2 - 9), 9 = \omega^2(A^2 - 16). \text{ Subtract: } 7 = 7\omega^2 \Rightarrow \omega = 1 \text{ rad/s.}$$

$$\text{Then } A^2 = 25 \Rightarrow A = 5 \text{ cm.}$$

Answer: $A = 5 \text{ cm}$; $T = 2\pi \text{ s} \approx 6.28 \text{ s}$.

Q3. What fraction of the total energy is kinetic when the displacement is half the amplitude?

$$KE/E = (A^2 - x^2)/A^2 = 1 - 1/4 = 3/4.$$



Answer: 75% kinetic, 25% potential.

Q4. A capacitor of 2 μF is charged to 100 V and discharged through a 5 mH coil of negligible resistance. Find the frequency and peak current.

$$\omega_0 = 1/\sqrt{5 \times 10^{-3} \times 2 \times 10^{-6}} = 1/\sqrt{10^{-8}} = 10^4 \text{ rad/s}; \nu = 1592 \text{ Hz.}$$

$$q_0 = CV = 2 \times 10^{-4} \text{ C}; I_0 = \omega_0 q_0 = 2 \text{ A.}$$

Answer: $\nu \approx 1.59 \text{ kHz}; I_0 = 2 \text{ A.}$

Q5. A damped oscillator loses 5% of its energy in each cycle. Find its Q-factor.

$$Q = 2\pi \times E/\Delta E = 2\pi/0.05.$$

Answer: $Q \approx 126.$

Q6. Show that for light damping the amplitude of the damped oscillator decays to 1/e of its initial value in time 2τ , where τ is the relaxation time.

$$\text{Amplitude} \propto e^{-\gamma t}. \text{ It becomes } 1/e \text{ when } \gamma t = 1 \Rightarrow t = 1/\gamma = 2(1/2\gamma) = 2\tau.$$

Answer: Shown.

Q7. For what value of R does a series LCR circuit with $L = 2 \text{ H}$ and $C = 2 \mu\text{F}$ become critically damped?

$$R_c = 2\sqrt{L/C} = 2\sqrt{2/(2 \times 10^{-6})} = 2\sqrt{10^6} = 2000 \Omega.$$

Answer: $R = 2 \text{ k}\Omega.$

Q8. A forced oscillator has $\omega_0 = 100 \text{ rad/s}$ and $Q = 25$. Find the bandwidth $\Delta\omega$ and the amplitude-resonance frequency.

$$\gamma = \omega_0/2Q = 2 \text{ s}^{-1}; \Delta\omega = 2\gamma = 4 \text{ rad/s.}$$

$$\omega_r = \sqrt{(\omega_0^2 - 2\gamma^2)} = \sqrt{(10000 - 8)} = 99.96 \text{ rad/s.}$$

Answer: $\Delta\omega = 4 \text{ rad/s}; \omega_r \approx 99.96 \text{ rad/s.}$

Q9. A 60 kg person in a car causes the car springs to compress by 3 cm. Find the natural frequency of the person-plus-spring system alone.

$$k = mg/x = 60 \times 9.8/0.03 = 19600 \text{ N/m. } \omega = \sqrt{k/m} = \sqrt{(326.7)} = 18.07 \text{ rad/s.}$$

Answer: $\nu \approx 2.88 \text{ Hz.}$

Q10. Find the period of a simple pendulum of length 1 m on the Moon ($g = 1.62 \text{ m/s}^2$).

$$T = 2\pi\sqrt{1/1.62} = 2\pi \times 0.786.$$

Answer: $T \approx 4.94 \text{ s.}$

1.19 MCQs: Module I

1. In SHM, acceleration is proportional to

(a) square of displacement (b) displacement (c) velocity (d) time

□ (b) — $a = -\omega^2 x.$



- 2.** The equation of SHM is
(a) $\frac{d^2x}{dt^2} + \omega^2 x = 0$ (b) $\frac{dx}{dt} + \omega x = 0$ (c) $\frac{d^2x}{dt^2} - \omega^2 x = 0$ (d) $\frac{d^2x}{dt^2} + \omega^2 x = 0$
☐ (d) — Restoring acceleration.
- 3.** The period of a spring-mass system is
(a) $2\pi\sqrt{m/k}$ (b) $2\pi\sqrt{k/m}$ (c) $\sqrt{m/k}$ (d) $2\pi mk$
☐ (a) — $\omega = \sqrt{k/m}$.
- 4.** Total energy of a body in SHM is proportional to
(a) $1/A$ (b) $\sqrt{A}$ (c) A (d) A^2
☐ (d) — $E = (1/2)kA^2$.
- 5.** In SHM the velocity is maximum at
(a) the mean position (b) everywhere (c) the extreme position (d) halfway
☐ (a) — KE is maximum at $x = 0$.
- 6.** KE equals PE in SHM at x equal to
(a) A (b) $A/\sqrt{2}$ (c) $A/4$ (d) $A/2$
☐ (b) — $A^2 - x^2 = x^2$.
- 7.** The frequency of an LC circuit is
(a) $\sqrt{L/C}/2\pi$ (b) $2\pi\sqrt{LC}$ (c) $1/LC$ (d) $1/(2\pi\sqrt{LC})$
☐ (d) — $\omega_0 = 1/\sqrt{LC}$.
- 8.** In the mechanical-electrical analogy, mass corresponds to
(a) inductance (b) capacitance (c) charge (d) resistance
☐ (a) — Inertia $\leftrightarrow$ inductance.
- 9.** The spring constant k corresponds to
(a) $1/C$ (b) L (c) R (d) C
☐ (a) — $PE = q^2/2C$.
- 10.** The damping force in a damped oscillator is usually taken proportional to
(a) displacement (b) velocity (c) acceleration (d) mass
☐ (b) — $F = -bv$.
- 11.** For heavy damping,
(a) $\gamma = \omega_0$ (b) $\gamma < \omega_0$ (c) $\gamma = 0$ (d) $\gamma > \omega_0$
☐ (d) — Over-damped.
- 12.** Critical damping occurs when
(a) $\gamma = \omega_0$ (b) $\gamma < \omega_0$ (c) $\gamma > \omega_0$ (d) $\omega_0 = 0$
☐ (a) — Fastest non-oscillatory return.
- 13.** Galvanometers are designed to be
(a) heavily damped (b) critically damped (c) lightly damped (d) undamped
☐ (b) — Quickest settling without oscillation.
- 14.** The frequency of a lightly damped oscillator is
(a) $\sqrt{\omega_0^2 + \gamma^2}$ (b) $\sqrt{\omega_0^2 - \gamma^2}$ (c) $\omega_0 - \gamma$ (d) ω_0
☐ (b) — $\omega' < \omega_0$.
- 15.** In light damping the amplitude decays as
(a) $e^{-\gamma t}$ (b) $1/t$ (c) $e^{-2\gamma t}$ (d) $e^{\gamma t}$
☐ (a) — $x = A_0 e^{-\gamma t} \cos(\omega' t + \phi)$.
- 16.** The energy of a lightly damped oscillator decays as
(a) $e^{-\gamma t}$ (b) $e^{-\gamma t/2}$ (c) linearly (d) $e^{-2\gamma t}$
☐ (d) — $E \propto \text{amplitude}^2$.



- 17.** The relaxation time τ is
(a) $1/\gamma$ (b) $1/(2\gamma)$ (c) $\gamma/2$ (d) 2γ
☐ (b) — $E = E_0 e^{-t/\tau}$.
- 18.** The quality factor Q equals
(a) γ/ω (b) $\omega\tau$ (c) ω/γ (d) τ/ω
☐ (b) — $Q = \omega\tau = \omega/2\gamma$.
- 19.** A high Q -factor means
(a) zero frequency (b) no oscillation (c) low damping (d) heavy damping
☐ (c) — Energy lost slowly.
- 20.** Q -factor of a series LCR circuit is
(a) $\omega_0 RC$ (b) $\omega_0 L/R$ (c) LC/R (d) $R/\omega_0 L$
☐ (b) — $Q = \omega_0 L/R$.
- 21.** The logarithmic decrement equals
(a) γT (b) γ/T (c) $\omega_0 T$ (d) T/γ
☐ (a) — $\ln(A_n/A_{n+1}) = \gamma T$.
- 22.** The critical resistance of an LCR circuit is
(a) LC (b) $\sqrt{L/C}$ (c) $2\sqrt{C/L}$ (d) $2\sqrt{L/C}$
☐ (d) — $\gamma = \omega_0 \Rightarrow R/2L = 1/\sqrt{LC}$.
- 23.** The number of oscillations for energy to fall to $1/e$ is
(a) Q^2 (b) Q (c) $2\pi Q$ (d) $Q/2\pi$
☐ (d) — $\nu\tau = Q/2\pi$.
- 24.** In forced oscillations, the steady-state frequency is
(a) the damped frequency (b) zero (c) the natural frequency (d) the driving frequency
☐ (d) — The system follows the driver.
- 25.** The transient solution in forced oscillations
(a) grows with time (b) dies out with time (c) equals the steady state (d) is constant
☐ (b) — Decays as $e^{-\gamma t}$.
- 26.** Amplitude resonance occurs at
(a) $\omega = \sqrt{(\omega_0^2 - 2\gamma^2)}$ (b) $\omega = 2\omega_0$ (c) $\omega = \gamma$ (d) $\omega = \omega_0$
☐ (a) — Slightly below ω_0 .
- 27.** Velocity resonance occurs at
(a) $\omega = \gamma$ (b) $\omega = \omega_0$ (c) $\omega = 0$ (d) $\omega = \sqrt{(\omega_0^2 - 2\gamma^2)}$
☐ (b) — Velocity in phase with force.
- 28.** At resonance, the phase difference between displacement and driving force is
(a) $\pi/4$ (b) $\pi/2$ (c) 0 (d) π
☐ (b) — $\tan \delta \rightarrow \infty$.
- 29.** At very high driving frequency, the displacement is
(a) zero phase (b) in phase with force (c) π behind (out of phase) (d) $\pi/2$ behind
☐ (c) — $\delta \rightarrow \pi$.
- 30.** At low driving frequency, the amplitude is controlled by
(a) frequency (b) mass (c) damping (d) stiffness
☐ (d) — $A \approx F_0/k$.
- 31.** At resonance, the amplitude is controlled by
(a) damping (b) stiffness (c) mass (d) none
☐ (a) — $A = F_0/(b\omega_0)$.



- 32.** Sharpness of resonance increases as damping
(a) decreases (b) increases (c) becomes infinite (d) is constant
□ (a) — $Q = \omega_0/2\gamma$.
- 33.** The half-power bandwidth $\Delta\omega$ for small damping equals
(a) $\gamma/2$ (b) γ (c) ω_0 (d) 2γ
□ (d) — $Q = \omega_0/\Delta\omega$.
- 34.** At series resonance in an LCR circuit, the impedance is
(a) minimum and equal to R (b) zero (c) infinite (d) maximum
□ (a) — $X_L = X_C$.
- 35.** Voltage magnification at series resonance equals
(a) $1/Q$ (b) R (c) LC (d) Q
□ (d) — $V_L = QV$.
- 36.** The phase difference between current and voltage at series resonance is
(a) 180° (b) 0° (c) 45° (d) 90°
□ (b) — Purely resistive.
- 37.** The Tacoma Narrows bridge collapse is an example of
(a) damping (b) diffraction (c) resonance (d) interference
□ (c) — Wind-driven resonance.
- 38.** If the mass of a spring-mass system is made four times, the period becomes
(a) 2 times (b) half (c) 4 times (d) same
□ (a) — $T \propto \sqrt{m}$.
- 39.** If the length of a simple pendulum is doubled, the period becomes
(a) $T/\sqrt{2}$ (b) $\sqrt{2} T$ (c) $T/2$ (d) $2T$
□ (b) — $T \propto \sqrt{l}$.
- 40.** The period of an LC circuit with $L = 1 \text{ H}$ and $C = 1 \text{ F}$ is
(a) $2\pi \text{ s}$ (b) $1/2\pi \text{ s}$ (c) $\pi \text{ s}$ (d) 1 s
□ (a) — $T = 2\pi\sqrt{LC}$.

Module II: Waves and Optics

A wave is a disturbance that carries energy and momentum from one place to another without any net transport of matter. Sound, light, radio signals, ripples on water and vibrations of strings are all waves. This module builds the wave equation and then uses the principle of superposition to explain interference and diffraction of light.

2.1 Concept of a wave

- **Transverse wave:** particles of the medium vibrate perpendicular to the direction of propagation (waves on a string, electromagnetic waves).
- **Longitudinal wave:** particles vibrate along the direction of propagation (sound in air).
- **Wavelength λ :** distance between two consecutive points in the same phase.
Frequency ν : number of oscillations per second. **Wave speed $v = \nu\lambda$.**
- **Wave number $k = 2\pi/\lambda$ (rad/m); angular frequency $\omega = 2\pi\nu$; so $v = \omega/k$.**

A progressive harmonic wave travelling in the +x direction is

$$y(x, t) = A \sin(\omega t - kx) = A \sin (2\pi/\lambda)(vt - x)$$

$$\text{Phase difference} = (2\pi/\lambda) \times \text{path difference; time difference} = T/(2\pi) \times \text{phase difference}$$

A wave travelling in the -x direction is $y = A \sin(\omega t + kx)$.

2.2 The one-dimensional wave equation

Any function of the form $y = f(x - vt)$ or $y = g(x + vt)$ represents a wave moving with speed v without change of shape. Differentiating twice:

$$\partial^2 y / \partial t^2 = v^2 \partial^2 y / \partial x^2 \text{ (one-dimensional wave equation)}$$

Derivation for a stretched string. Consider a small element of a string of mass per unit length μ under tension T . The tension acts along the string at angles θ_1 and θ_2 at the two ends. Net vertical force = $T(\sin \theta_2 - \sin \theta_1) \approx T(\tan \theta_2 - \tan \theta_1) = T[(\partial y / \partial x)_{x+dx} - (\partial y / \partial x)_x] = T(\partial^2 y / \partial x^2)dx$. By Newton's law this equals $(\mu dx)(\partial^2 y / \partial t^2)$. Hence $\partial^2 y / \partial t^2 = (T/\mu) \partial^2 y / \partial x^2$, so the wave speed on a string is

$$v = \sqrt{T/\mu}$$

Similarly, sound in a gas has $v = \sqrt{\gamma P / \rho}$, and light in vacuum has $c = 1/\sqrt{\mu_0 \epsilon_0}$ (derived in Module III).

Three-dimensional wave equation: $\nabla^2 \psi = (1/v^2) \partial^2 \psi / \partial t^2$, where $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$.

2.3 Principle of superposition

When two or more waves overlap, the resultant displacement at any point is the algebraic (vector) sum of the individual displacements: $y = y_1 + y_2 + \dots$. This holds because the wave equation is linear.

Superposition of two waves of the same frequency. Let $y_1 = a_1 \sin \omega t$ and $y_2 = a_2 \sin(\omega t + \delta)$. The resultant is $y = R \sin(\omega t + \theta)$ with

$$R^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos \delta, \tan \theta = a_2 \sin \delta / (a_1 + a_2 \cos \delta)$$

$$\text{Intensity } I \propto R^2: I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \delta$$

- **Constructive interference:** $\delta = 0, 2\pi, 4\pi, \dots$ (path difference $= n\lambda$) $\Rightarrow R = a_1 + a_2$, maximum intensity.
- **Destructive interference:** $\delta = \pi, 3\pi, \dots$ (path difference $= (2n + 1)\lambda/2$) $\Rightarrow R = |a_1 - a_2|$, minimum intensity.
- For equal amplitudes a : $I = 4a^2 \cos^2(\delta/2)$; $I_{\max} = 4a^2$ and $I_{\min} = 0$.

Superposition of many harmonic waves (n waves with equal amplitude a and successive phase difference δ). Adding the phasors gives

$$R = a \sin(n\delta/2)/\sin(\delta/2)$$

This result is the basis of the diffraction grating (Section 2.10).

Energy conservation. Interference does not destroy energy; it only redistributes it. The average intensity over the pattern equals $I_1 + I_2$.

2.4 Coherent sources

Two sources are **coherent** if they emit waves of the same frequency (wavelength) with a constant phase difference (or zero). Only coherent sources produce a stable interference pattern.

Two independent light sources (two bulbs) are never coherent, because each atom emits random, short wave trains (about 10^{-8} s long) with random phases. So coherent sources are always derived from a single source by one of two methods:

- **Division of wavefront:** the wavefront from one source is divided into two parts by slits or mirrors. Examples: Young's double slit, Fresnel's biprism, Lloyd's mirror.
- **Division of amplitude:** the amplitude of a single wave is divided by partial reflection and partial transmission. Examples: thin films, Newton's rings, Michelson interferometer.

Conditions for sustained interference: coherent sources, same frequency, nearly equal amplitudes (for good contrast), narrow sources, small separation between sources, and a large distance to the screen.

2.5 Interference in thin parallel films (reflected light)

Consider a thin film of thickness t and refractive index μ in air. A ray is partly reflected from the top surface and partly refracted, reflected at the bottom surface and emerges. The two reflected rays are coherent (division of amplitude).

Optical path difference. Geometry (with angle of refraction r inside the film) gives the path difference $2\mu t \cos r$. In addition, reflection at the top surface (from a rarer to a denser medium) introduces a phase change of π , equivalent to an extra path of $\lambda/2$.

$$\text{Effective path difference } \Delta = 2\mu t \cos r - \lambda/2$$

Reflected light — Bright (maxima): $2\mu t \cos r = (2n - 1)\lambda/2$; Dark (minima): $2\mu t \cos r = n\lambda$, $n = 0, 1, 2, \dots$

Transmitted light — conditions are reversed: Bright: $2\mu t \cos r = n\lambda$; Dark: $2\mu t \cos r = (2n - 1)\lambda/2$

- The reflected and transmitted patterns are **complementary**: a film that looks bright in reflection for a wavelength looks dark in transmission for that wavelength.

- A very thin film ($t \rightarrow 0$) appears **dark** in reflected light, because the path difference is $\lambda/2$ only.
- Colours of soap bubbles and oil films on water arise because the condition for brightness is satisfied for different wavelengths at different thicknesses and viewing angles.
- **Anti-reflection coating:** a film with $\mu t = \lambda/4$ (quarter-wave) on a lens makes the reflected rays cancel.

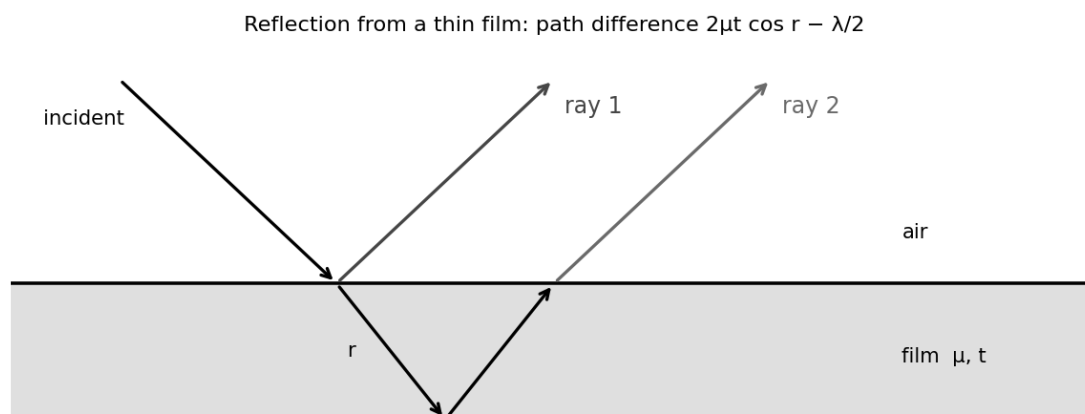


Figure 2.1: Two coherent reflected rays from a thin film (division of amplitude).

2.6 Wedge-shaped film

A thin air wedge of small angle θ between two glass plates, viewed normally with monochromatic light, shows straight, equally spaced fringes parallel to the edge.

$$\text{Fringe width } \beta = \lambda/(2\mu\theta) \text{ (for air } \mu = 1: \beta = \lambda/2\theta)$$

At the edge ($t = 0$), the fringe is dark. Wedge fringes are used to test the flatness of surfaces and to measure the thickness of thin wires or papers (thickness = $N\lambda/2$ for N dark fringes).

2.7 Newton's rings

Formation. A plano-convex lens of large radius of curvature R is placed on a flat glass plate. A thin air film of gradually increasing thickness is formed between them. When monochromatic light falls normally, reflected light shows alternate dark and bright **concentric circular rings** around the point of contact. The rings are circular because points of equal film thickness lie on circles centred at the contact point (fringes of equal thickness).

Theory. Let a ring of radius r be at a point where the air film thickness is t . From the geometry of the circle, $r^2 = (2R - t)t \approx 2Rt$ (since $t \ll R$), so

$$t = r^2/(2R)$$

For normal incidence ($\cos r = 1$) and an air film ($\mu = 1$), the condition for a dark ring in reflected light is $2t = n\lambda$. Substituting:

$$\text{Dark rings: } r_n^2 = n\lambda R, D_n^2 = 4n\lambda R \text{ (D = diameter)}$$

$$\text{Bright rings: } r_n^2 = (2n - 1)\lambda R/2, D_n^2 = 2(2n - 1)\lambda R$$

Newton's rings (reflected light)

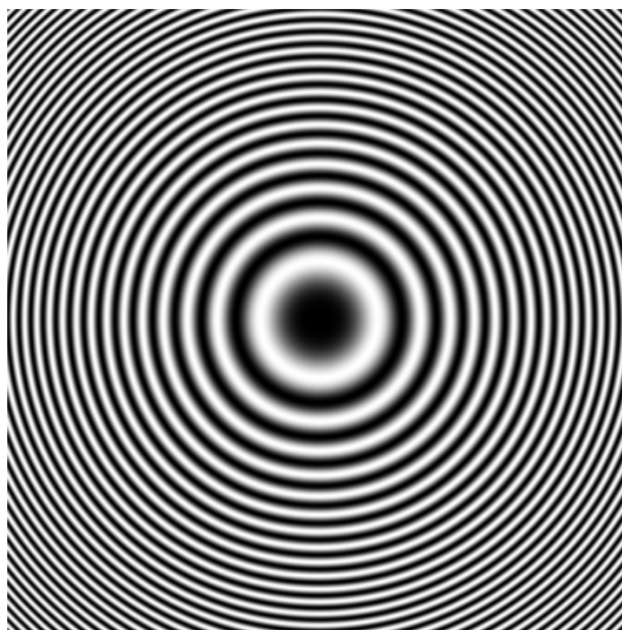


Figure 2.2: Newton's rings in reflected light. The centre is dark and the rings get closer together outward ($D \propto \sqrt{n}$).

Important features.

- The central spot ($t = 0$) is **dark** in reflected light because of the phase change of π .
- $D_n \propto \sqrt{n}$ for dark rings, so the rings get closer together as n increases (they are not equally spaced).
- In transmitted light the central spot is bright and the pattern is complementary.
- If a liquid of refractive index μ fills the gap: $D_n^2 = 4n\lambda R/\mu$, so the rings contract.

Determination of the wavelength of light. In practice the exact order at the centre is uncertain (dust, deformation), so we measure the diameters of two rings, the n -th and the $(n + p)$ -th:

$$D_{n+p}^2 - D_n^2 = 4p\lambda R \Rightarrow \lambda = (D_{n+p}^2 - D_n^2)/(4pR)$$

The radius of curvature R is measured with a spherometer. A graph of D_n^2 against n is a straight line of slope $4\lambda R$.

Determination of the refractive index of a liquid. Measure $(D_{n+p}^2 - D_n^2)$ first with air and then with the liquid filling the gap:

$$\mu = (D_{n+p}^2 - D_n^2)_{\text{air}} / (D_{n+p}^2 - D_n^2)_{\text{liquid}}$$

Experimental arrangement. Light from a sodium lamp ($\lambda = 589.3 \text{ nm}$) is rendered parallel by a convex lens and falls on a glass plate inclined at 45° , which reflects it vertically down onto the lens-plate combination. The rings are viewed with a travelling microscope placed above. (See Physics Lab, Experiment 2.)

2.8 Diffraction and Huygens' principle

Diffraction is the bending of light around the edges of an obstacle or aperture and its spreading into the region of geometrical shadow. It is noticeable when the size of the

obstacle or aperture is comparable to the wavelength.

Huygens' principle. (1) Every point on a wavefront acts as a source of secondary wavelets that spread out in all directions with the speed of the wave. (2) The new wavefront at a later time is the forward envelope (common tangent) of these secondary wavelets. Fresnel added that the secondary wavelets interfere with one another; the Huygens-Fresnel principle explains diffraction patterns.

Interference versus diffraction.

Interference	Diffraction
Superposition of waves from two (or a few) coherent sources	Superposition of secondary wavelets from different parts of the same wavefront
Fringes are usually of equal width	Fringes are of unequal width (central maximum is widest)
All bright fringes have nearly the same intensity	Intensity of successive maxima decreases rapidly
Dark fringes are perfectly dark	Minima are not perfectly dark in general

2.9 Types of diffraction

Fresnel diffraction	Fraunhofer diffraction
Source and screen at finite distances from the aperture	Source and screen effectively at infinite distance (lenses are used)
Incident wavefront is spherical or cylindrical	Incident wavefront is plane
No lenses needed	Converging lenses are used
Mathematical treatment is complicated (Fresnel zones)	Mathematical treatment is simple
Examples: straight edge, circular aperture near a source	Examples: single slit, double slit, grating with a spectrometer

2.10 Fraunhofer diffraction at a single slit

A plane wavefront of wavelength λ falls normally on a narrow slit of width a . The secondary wavelets from all points of the slit reach a point P on the screen (in direction θ) with different phases. The path difference between the wavelets from the two edges of the slit is $a \sin \theta$.

Result (by adding the contributions of all points of the slit).

$$I = I_0 (\sin \beta / \beta)^2, \quad \beta = (\pi a \sin \theta) / \lambda$$

Positions.

Central maximum: $\theta = 0$ (all wavelets in phase)

Minima: $a \sin \theta = n\lambda$, $n = \pm 1, \pm 2, \dots$

Secondary maxima (approx.): $a \sin \theta = (2n + 1)\lambda/2$, intensities $\approx I_0/22, I_0/62, \dots$ (4.5%, 1.6%, ...)

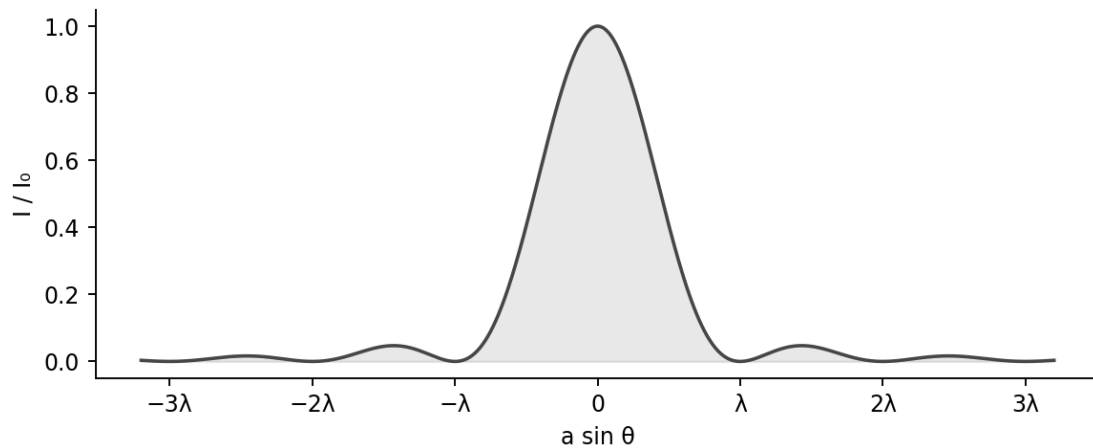


Figure 2.3: Intensity distribution in single-slit Fraunhofer diffraction, $I = I_0(\sin \beta/\beta)^2$.

Half-zone explanation of the first minimum. Divide the slit into two halves. For a $\sin \theta = \lambda$, each point in the upper half has a partner in the lower half at a path difference $\lambda/2$. All such pairs cancel, giving darkness.

Width of the central maximum. Angular half-width $\theta \approx \lambda/a$. On a screen at distance D (or at the focal plane of a lens of focal length f), linear width $= 2\lambda D/a$ (or $2\lambda f/a$). A narrower slit gives a wider central maximum.

2.11 Diffraction grating (qualitative and basic theory)

A **plane transmission grating** is a glass plate on which a very large number of equidistant parallel lines (opaque rulings) are drawn. The transparent spaces act as slits. If a is the width of each slit and b the width of each opaque line, then $(a + b) = d$ is the **grating element** (grating spacing). If there are N lines per metre, $d = 1/N$.

Grating equation. Light from corresponding points of successive slits has path difference $(a + b) \sin \theta$. Principal maxima (bright lines) occur when all slits reinforce:

$$(a + b) \sin \theta = n\lambda, n = 0, 1, 2, \dots \text{ (order of the spectrum)}$$

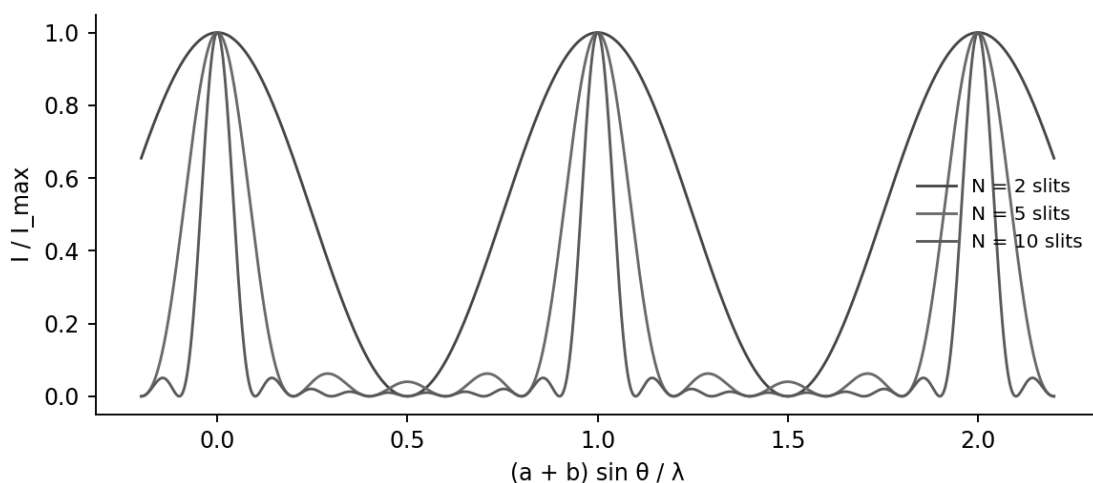


Figure 2.4: As the number of slits N increases, the principal maxima become sharper and $N - 2$ weak secondary maxima appear between them.

Features.

- With N slits, each principal maximum is very sharp and bright; its intensity $\propto N^2$. Between principal maxima there are $N - 1$ minima and $N - 2$ weak secondary maxima.
- White light is split into a spectrum in each order (violet deviated least, red most). The zero-order ($n = 0$) image is white.
- **Maximum possible order:** since $\sin \theta \leq 1$, $n_{\max} = (a + b)/\lambda$ (take the integer part).
- **Absent spectra:** if a $\sin \theta = m\lambda$ (a single-slit minimum) at the same angle as a grating maximum, that order is missing. Condition: $(a + b)/a = n/m$.
- **Dispersive power** $d\theta/d\lambda = n/((a + b) \cos \theta)$. **Resolving power** $R = \lambda/d\lambda = nN$ (qualitative: more lines and higher order give better resolution).

Determination of the grating element / wavelength with a spectrometer. The spectrometer is adjusted for parallel light, the grating is set normal to the incident beam, and the angle 2θ between the first-order images on the two sides is measured. Then $(a + b) = n\lambda/\sin \theta$. (See Physics Lab, Experiment 3.)

2.12 Common mistakes to avoid in Module II

- Forgetting the extra $\lambda/2$ due to reflection at a denser medium in thin-film problems.
- Using radius instead of diameter (or vice versa) in Newton's ring formulas. $D^2 = 4r^2$.
- Writing the single-slit minimum condition as a $\sin \theta = (2n + 1)\lambda/2$. That is (approximately) the secondary maximum; minima are at a $\sin \theta = n\lambda$.
- Taking N (lines per cm) directly as $(a + b)$. The grating element is $(a + b) = 1/N$, with units converted to metres.
- Giving a non-integer order. The number of orders is the integer part of $(a + b)/\lambda$.

2.13 Long-answer derivations (exam style)

(A) Path difference in a thin film (reflected light). Let a ray AB be incident at angle i on a film of thickness t and index μ . It is partly reflected along BC (ray 1) and partly refracted along BD at angle r ; at D it is reflected to E and emerges along EF (ray 2), parallel to BC . Draw EG perpendicular to BC . The optical path difference is $\Delta = \mu(BD + DE) - BG$.

- $BD = DE = t/\cos r$, so $\mu(BD + DE) = 2\mu t/\cos r$.
- $BE = 2t \tan r$, and $BG = BE \sin i = 2t \tan r \sin i = 2t \tan r (\mu \sin r)$ (Snell's law $\sin i = \mu \sin r$).
- $\Delta = 2\mu t/\cos r - 2\mu t \sin^2 r/\cos r = 2\mu t(1 - \sin^2 r)/\cos r = 2\mu t \cos r$.

Ray 1 is reflected from a denser medium and undergoes a phase change of π (Stokes' relation), which adds $\lambda/2$. So the effective path difference is $2\mu t \cos r - \lambda/2$, leading to the conditions in Section 2.5.

Transmitted light. The two transmitted rays have the same geometrical path difference $2\mu t \cos r$ but no net phase change (both internal reflections are from a rarer medium). So the conditions are reversed.

(B) Newton's rings: relation between radius and thickness. Let R be the radius of curvature, O the point of contact, and consider a point at distance r from O where the air film has thickness t . From the property of intersecting chords of a circle, $r \times r = t \times (2R - t)$. Since $t \ll R$, $r^2 \approx 2Rt$, i.e. $t = r^2/2R$.

Dark ring condition (normal incidence, air film, reflected light): $2t = n\lambda = r^2/R = n\lambda = r_n^2 = n\lambda R$, $D_n^2 = 4n\lambda R$.

Bright ring condition: $2t = (2n - 1)\lambda/2 \Rightarrow D_n^2 = 2(2n - 1)\lambda R$.

Spacing: $D_{n+1} - D_n = 2\sqrt{(\lambda R)(\sqrt{n+1} - \sqrt{n})} \approx \sqrt{(\lambda R/n)}$, which decreases as n increases.

(C) Intensity in single-slit diffraction (phasor method). Divide the slit of width a into N equal strips. The phase difference between wavelets from adjacent strips reaching P is $\delta = (2\pi/\lambda)(a/N) \sin \theta$. The total phase difference across the slit is $N\delta = 2\beta$, where $\beta = \pi a \sin \theta/\lambda$. The resultant of N equal phasors of amplitude A_0/N arranged along an arc of total angle 2β is a chord: $R = A_0 \sin \beta/\beta$ (as $N \rightarrow \infty$). Hence $I = I_0(\sin \beta/\beta)^2$.

- Maximum at $\beta = 0$ (central maximum).
- Minima where $\sin \beta = 0$, $\beta \neq 0$: $\beta = n\pi \Rightarrow a \sin \theta = n\lambda$.
- Secondary maxima where $d/d\beta(\sin \beta/\beta) = 0 \Rightarrow \tan \beta = \beta \Rightarrow \beta \approx 1.43\pi, 2.46\pi, 3.47\pi, \dots$
Intensities: $I_0(\sin \beta/\beta)^2 \approx 0.047I_0, 0.016I_0, 0.008I_0$.

(D) Intensity for a grating (N slits). Each slit contributes the single-slit amplitude $A_0 \sin \beta/\beta$. The N slits have successive phase difference 2γ , where $\gamma = \pi(a + b) \sin \theta/\lambda$. Adding N such phasors (Section 2.3): $I = I_0(\sin \beta/\beta)^2(\sin N\gamma/\sin \gamma)^2$.

- $(\sin \beta/\beta)^2$ is the single-slit diffraction envelope.
- $(\sin N\gamma/\sin \gamma)^2$ gives sharp principal maxima of height N^2 when $\gamma = n\pi$, i.e. $(a + b) \sin \theta = n\lambda$, the grating equation.
- Minima occur when $N\gamma = m\pi$ (m not a multiple of N), giving $N - 1$ minima between principal maxima.

2.14 Experimental notes (link to Physics Lab)

Newton's rings (Lab Expt. 2). Clean the lens and plate; focus the microscope on the rings; place the cross-wire tangentially on, say, the 20th ring on the left and move inwards, noting readings for rings 20, 18, 16, ..., 4, then continue across the centre to the right side. Diameter = difference of left and right readings. Plot D^2 vs n ; slope = $4\lambda R$. Measure R with a spherometer: $R = l^2/(6h) + h/2$ (l = mean distance between legs, h = sagitta). Sources of error: backlash of the microscope screw (always move in one direction), dust at the contact point, counting mistakes.

Grating with spectrometer (Lab Expt. 3). Adjust the eyepiece (cross-wires sharp), the telescope (for parallel rays, focused on a distant object), the collimator (slit sharp and narrow), and level the prism table. Set the grating normal to the incident light (rotate the telescope by 90° from the direct image, set the grating to reflect the slit onto the cross-wires, then rotate the table by 45°). Measure the angles of first-order images on both sides: $\theta = (\text{difference})/2$. Then $(a + b) = n\lambda/\sin \theta$, with $\lambda = 5893 \text{ \AA}$ for sodium (or known mercury lines).

2.15 Comparison: interference by division of wavefront vs division of amplitude

Division of wavefront	Division of amplitude
The wavefront is split into two parts by apertures, mirrors or prisms	The amplitude is split by partial reflection and transmission
Needs a narrow (point or slit) source	Works with broad extended sources
Young's double slit, Fresnel's biprism, Lloyd's mirror	Thin films, Newton's rings, Michelson interferometer



Division of wavefront	Division of amplitude
Fringes are usually faint	Fringes are bright

2.16 Solved numericals: Module II

Problem 1. The equation of a wave is $y = 0.05 \sin(100\pi t - 2\pi x)$ (SI units). Find the amplitude, frequency, wavelength and speed.

Compare with $y = A \sin(\omega t - kx)$: $A = 0.05$ m, $\omega = 100\pi \Rightarrow \nu = 50$ Hz, $k = 2\pi \Rightarrow \lambda = 1$ m.

$v = \nu\lambda = 50$ m/s (also $v = \omega/k = 100\pi/2\pi = 50$ m/s).

Answer: $A = 5$ cm, $\nu = 50$ Hz, $\lambda = 1$ m, $v = 50$ m/s.

Problem 2. A string of mass per unit length 5 g/m is under a tension of 80 N. Find the speed of transverse waves on it.

$v = \sqrt{T/\mu} = \sqrt{(80/0.005)} = \sqrt{16000}$.

Answer: $v \approx 126.5$ m/s.

Problem 3. Two waves of equal amplitude a and phase difference $\pi/3$ superpose. Find the resultant amplitude and intensity relative to one wave.

$R^2 = a^2 + a^2 + 2a^2 \cos 60^\circ = 3a^2 \Rightarrow R = \sqrt{3} a$.

Answer: $R = 1.73a$; intensity = 3 times that of one wave.

Problem 4. Two coherent sources have intensity ratio $9 : 1$. Find the ratio of maximum to minimum intensity in the interference pattern.

$a_1/a_2 = \sqrt{9} = 3$. $I_{\max}/I_{\min} = (a_1 + a_2)^2/(a_1 - a_2)^2 = (4/2)^2 = 4$.

Answer: $I_{\max} : I_{\min} = 4 : 1$.

Problem 5. A soap film of refractive index 1.33 and thickness $5000 \text{ \AA}$ is viewed at normal incidence in white light. Which wavelengths in the visible range (4000 - $7000 \text{ \AA}$) are strongly reflected?

Bright in reflection: $2\mu t = (2n - 1)\lambda/2 \Rightarrow \lambda = 4\mu t/(2n - 1) = 4 \times 1.33 \times 5000/(2n - 1) = 26600/(2n - 1) \text{ \AA}$.

$n = 3$: $\lambda = 5320 \text{ \AA}$ (visible). $n = 4$: $\lambda = 3800 \text{ \AA}$ (UV). $n = 2$: $\lambda = 8867 \text{ \AA}$ (IR).

Answer: Only $\lambda = 5320 \text{ \AA}$ (green) is strongly reflected.

Problem 6. A parallel beam of light ($\lambda = 5890 \text{ \AA}$) is incident on a thin glass plate ($\mu = 1.5$) such that the angle of refraction is 60° . Find the smallest thickness that makes the plate appear dark in reflected light.

Dark: $2\mu t \cos r = n\lambda$. Smallest non-zero thickness: $n = 1$.

$t = \lambda/(2\mu \cos r) = 5890 \times 10^{-10}/(2 \times 1.5 \times 0.5) = 5890 \times 10^{-10}/1.5$.

Answer: $t \approx 3.93 \times 10^{-7} \text{ m} = 3927 \text{ \AA}$.

Problem 7. An oil film ($\mu = 1.4$) on water appears bright green ($\lambda = 5600 \text{ \AA}$) at normal incidence in the first order. Find its minimum thickness.

Assume the film is between air and water ($\mu_{\text{water}} = 1.33 < 1.4$), so there is one phase reversal. Bright: $2\mu t = \lambda/2$ ($n = 1$).

$$t = \lambda/(4\mu) = 5600/(5.6) \text{ \AA}.$$

Answer: $t = 1000 \text{ \AA} = 100 \text{ nm}$.

Problem 8. Calculate the thickness of the anti-reflection (quarter-wave) coating of MgF_2 ($\mu = 1.38$) for $\lambda = 550 \text{ nm}$.

$$t = \lambda/(4\mu) = 550/(4 \times 1.38) \text{ nm}.$$

Answer: $t \approx 99.6 \text{ nm}$.

Problem 9. An air wedge is formed between two glass plates touching at one end and separated by a wire at the other end, 10 cm away. With $\lambda = 6000 \text{ \AA}$, 20 fringes are seen per cm. Find the wedge angle and the diameter of the wire.

$$\text{Fringe width } \beta = 1 \text{ cm}/20 = 0.05 \text{ cm} = 5 \times 10^{-4} \text{ m}.$$

$$\theta = \lambda/(2\beta) = 6 \times 10^{-7}/(10^{-3}) = 6 \times 10^{-4} \text{ rad}.$$

$$\text{Diameter } d = \theta \times 0.1 \text{ m} = 6 \times 10^{-5} \text{ m}.$$

Answer: $\theta = 6 \times 10^{-4} \text{ rad}$; $d = 0.06 \text{ mm}$.

Problem 10. In a Newton's rings experiment, the diameter of the 5th dark ring is 0.336 cm and that of the 15th dark ring is 0.590 cm. The radius of curvature of the lens is 100 cm. Find the wavelength of light.

$$\lambda = (D_{15}^2 - D_5^2)/(4pR), p = 10, R = 1 \text{ m}.$$

$$D_{15}^2 = (0.590 \times 10^{-2})^2 = 3.481 \times 10^{-5} \text{ m}^2. D_5^2 = (0.336 \times 10^{-2})^2 = 1.129 \times 10^{-5} \text{ m}^2. \text{ Difference} = 2.352 \times 10^{-5} \text{ m}^2.$$

$$\lambda = 2.352 \times 10^{-5}/(4 \times 10 \times 1) = 5.88 \times 10^{-7} \text{ m}.$$

Answer: $\lambda \approx 5880 \text{ \AA}$.

Problem 11. In Newton's rings with light of wavelength $5890 \text{ \AA}$, the diameter of the 10th dark ring is 0.5 cm. Find the radius of curvature of the lens and the thickness of the air film at that ring.

$$D_{10}^2 = 4n\lambda R \Rightarrow R = D^2/(4n\lambda) = (0.5 \times 10^{-2})^2/(4 \times 10 \times 5890 \times 10^{-10}) = 2.5 \times 10^{-5}/(2.356 \times 10^{-5}).$$

$$R \approx 1.061 \text{ m}. \text{ Film thickness } t = n\lambda/2 = 10 \times 5890 \times 10^{-10}/2 = 2.945 \times 10^{-6} \text{ m}.$$

Answer: $R \approx 1.06 \text{ m}$; $t \approx 2.95 \text{ }\mu\text{m}$.

Problem 12. The diameter of the 10th dark ring changes from 1.40 cm to 1.27 cm when a liquid is introduced between the lens and the plate. Find the refractive index of the liquid.

$$\mu = D_{\text{air}}^2/D_{\text{liquid}}^2 = (1.40)^2/(1.27)^2 = 1.96/1.6129.$$

Answer: $\mu \approx 1.215$.

Problem 13. In Newton's rings the diameters of the 4th and 12th dark rings are 0.400 cm and 0.700 cm. Find the diameter of the 20th dark ring.

$$D_{12}^2 - D_4^2 = 8(4\lambda R) = 0.49 - 0.16 = 0.33 \text{ cm}^2. \text{ So } 4\lambda R = 0.04125 \text{ cm}^2.$$

$$D_{20}^2 = D_{12}^2 + 8(4\lambda R) = 0.49 + 0.33 = 0.82 \text{ cm}^2.$$

Answer: $D_{20} \approx 0.906 \text{ cm}$.

Problem 14. A single slit of width 0.2 mm is illuminated by light of wavelength 5000 Å. Find the angular position of the first minimum and the width of the central maximum on a screen 1 m away.

$$\sin \theta = \lambda/a = 5 \times 10^{-7}/(2 \times 10^{-4}) = 2.5 \times 10^{-3} \Rightarrow \theta \approx 2.5 \times 10^{-3} \text{ rad } (0.143^\circ).$$

$$\text{Width} = 2\lambda D/a = 2 \times 5 \times 10^{-7} \times 1/(2 \times 10^{-4}) = 5 \times 10^{-3} \text{ m}.$$

Answer: $\theta = 2.5 \times 10^{-3} \text{ rad}$; central maximum width = 5 mm.

Problem 15. In a single-slit Fraunhofer diffraction, the first minimum for red light (6600 Å) coincides with the first maximum (secondary) for another wavelength λ' . Find λ' .

First minimum of red: $a \sin \theta = 6600 \text{ Å}$. First secondary maximum of λ' : $a \sin \theta \approx 3\lambda'/2$.

$$3\lambda'/2 = 6600 \Rightarrow \lambda' = 4400 \text{ Å}.$$

Answer: $\lambda' = 4400 \text{ Å}$.

Problem 16. A plane transmission grating has 5000 lines per cm. Find the angle of diffraction of the second-order spectrum for $\lambda = 5460 \text{ Å}$ and the maximum order visible.

$$(a + b) = 1/5000 \text{ cm} = 2 \times 10^{-4} \text{ cm} = 2 \times 10^{-6} \text{ m}.$$

$$\sin \theta = n\lambda/(a + b) = 2 \times 5.46 \times 10^{-7}/(2 \times 10^{-6}) = 0.546 \Rightarrow \theta = 33.1^\circ.$$

$$n_{\max} = (a + b)/\lambda = 2 \times 10^{-6}/5.46 \times 10^{-7} = 3.66 \Rightarrow 3.$$

Answer: $\theta \approx 33.1^\circ$; maximum order = 3.

Problem 17. Light of wavelength 6000 Å falls normally on a grating. The first-order maximum is at 20° . Find the number of lines per cm.

$$(a + b) = \lambda/\sin 20^\circ = 6 \times 10^{-7}/0.342 = 1.754 \times 10^{-6} \text{ m} = 1.754 \times 10^{-4} \text{ cm}.$$

$$N = 1/(a + b) = 5701 \text{ lines/cm}.$$

Answer: About 5700 lines per cm.

Problem 18. A grating has 15000 lines per inch. Find the angular separation of the sodium D-lines (5890 Å and 5896 Å) in the first order.

$$(a + b) = 2.54/15000 \text{ cm} = 1.693 \times 10^{-6} \text{ m}.$$

$$\sin \theta_1 = 5.890 \times 10^{-7}/1.693 \times 10^{-6} = 0.3479 \Rightarrow \theta_1 = 20.355^\circ. \sin \theta_2 = 5.896 \times 10^{-7}/1.693 \times 10^{-6} = 0.3482 \Rightarrow \theta_2 = 20.377^\circ.$$

Answer: $\Delta\theta \approx 0.022^\circ \approx 1.3'$ (arc minutes).

Problem 19. How many lines must a grating have in order to resolve the sodium D-lines in the first order?

$$R = \lambda/d\lambda = 5893/6 \approx 982 = nN. \text{ For } n = 1: N \geq 982.$$

Answer: At least about 982 lines (illuminated).

2.17 More solved numericals: Module II

Problem 1. Waves from two coherent sources of equal amplitude meet at a point with a path difference of $\lambda/3$. Find the ratio of the intensity there to the maximum intensity.

$$\delta = (2\pi/\lambda)(\lambda/3) = 2\pi/3. I = 4I_0 \cos^2(\delta/2) = 4I_0 \cos^2(60^\circ) = I_0.$$

$$I_{\max} = 4I_0.$$

Answer: $I/I_{\max} = 1/4$.

Problem 2. White light is reflected normally from a soap film ($\mu = 1.33$) of thickness 3.8×10^{-5} cm. Which wavelengths are absent in the reflected light?

$$\text{Dark in reflection: } 2\mu t = n\lambda \Rightarrow \lambda = 2 \times 1.33 \times 3.8 \times 10^{-7}/n = 1.0108 \times 10^{-6}/n \text{ m.}$$

$$n = 2: 5054 \text{ \AA (visible). } n = 1: 10108 \text{ \AA (IR); } n = 3: 3369 \text{ \AA (UV).}$$

Answer: 5054 \AA is absent.

Problem 3. A thin film of oil ($\mu = 1.5$) of thickness $0.5 \mu\text{m}$ is seen in reflected light at an angle of incidence 45° . Find the wavelengths in the visible range that interfere constructively.

$$\sin r = \sin 45^\circ/1.5 = 0.4714 \Rightarrow \cos r = 0.8819. 2\mu t \cos r = 2 \times 1.5 \times 0.5 \times 10^{-6} \times 0.8819 = 1.3229 \times 10^{-6} \text{ m.}$$

$$\text{Bright: } 2\mu t \cos r = (2n - 1)\lambda/2 \Rightarrow \lambda = 2.6458 \times 10^{-6}/(2n - 1).$$

$$n = 3: 5292 \text{ \AA; } n = 4: 3780 \text{ \AA (UV); } n = 2: 8819 \text{ \AA (IR).}$$

Answer: 5292 \AA (green).

Problem 4. An air wedge between two glass plates, 15 cm long, is formed by a wire at one end. With $\lambda = 5890 \text{ \AA}$, 30 dark fringes are observed per cm. Find the wire diameter.

$$\beta = 1/30 \text{ cm} = 3.33 \times 10^{-4} \text{ m. } \theta = \lambda/2\beta = 5.89 \times 10^{-7}/(6.67 \times 10^{-4}) = 8.83 \times 10^{-4} \text{ rad.}$$

$$d = \theta \times 0.15 = 1.32 \times 10^{-4} \text{ m.}$$

Answer: $d \approx 0.132 \text{ mm}$.

Problem 5. Newton's rings are formed with $\lambda = 6000 \text{ \AA}$. The diameter of the 10th dark ring is 0.5 cm. Find the diameter of the 20th dark ring and the radius of curvature of the lens.

$$D_{20} = D_{10}\sqrt{2} = 0.707 \text{ cm.}$$

$$R = D_{10}^2/(4 \times 10 \times \lambda) = (5 \times 10^{-3})^2/(40 \times 6 \times 10^{-7}) = 2.5 \times 10^{-5}/2.4 \times 10^{-5}.$$



Answer: $D_{20} \approx 0.707$ cm; $R \approx 1.04$ m.

Problem 6. In a Newton's rings experiment the diameters of the 4th and 12th dark rings are 0.300 cm and 0.528 cm. If $R = 1$ m, find λ .

$$D_{12}^2 - D_4^2 = (0.528)^2 - (0.300)^2 = 0.2788 - 0.0900 = 0.1888 \text{ cm}^2 = 1.888 \times 10^{-5} \text{ m}^2.$$

$$\lambda = 1.888 \times 10^{-5} / (4 \times 8 \times 1) = 5.90 \times 10^{-7} \text{ m}.$$

Answer: $\lambda \approx 5900 \text{ \AA}$ (sodium light).

Problem 7. Newton's rings are observed with light of $5893 \text{ \AA}$ and a lens of $R = 100$ cm. If the space between lens and plate is filled with a liquid ($\mu = 1.33$), find the diameter of the 10th dark ring.

$$D_n^2 = 4n\lambda R/\mu = 4 \times 10 \times 5.893 \times 10^{-7} \times 1/1.33 = 1.772 \times 10^{-5} \text{ m}^2.$$

Answer: $D_{10} \approx 4.21$ mm.

Problem 8. Find the half angular width of the central maximum in single-slit diffraction for a slit of width 12×10^{-5} cm with light of $6000 \text{ \AA}$.

$$\sin \theta = \lambda/a = 6 \times 10^{-7} / 1.2 \times 10^{-6} = 0.5 \Rightarrow \theta = 30^\circ.$$

Answer: 30° .

Problem 9. Light of wavelength $5000 \text{ \AA}$ falls normally on a slit of width 0.1 mm. A lens of focal length 50 cm focuses the pattern. Find the distance between the first and second minima on one side.

$$y_n = n\lambda f/a. \text{ Spacing} = \lambda f/a = 5 \times 10^{-7} \times 0.5/10^{-4} = 2.5 \times 10^{-3} \text{ m}.$$

Answer: 2.5 mm.

Problem 10. A grating has 4000 lines per cm. At what angles will the first- and second-order spectra of the $4861 \text{ \AA}$ (blue) hydrogen line appear?

$$(a + b) = 2.5 \times 10^{-6} \text{ m}. \sin \theta_1 = 4.861 \times 10^{-7} / 2.5 \times 10^{-6} = 0.1944 \Rightarrow \theta_1 = 11.2^\circ.$$

$$\sin \theta_2 = 0.3889 \Rightarrow \theta_2 = 22.9^\circ.$$

Answer: 11.2° and 22.9° .

Problem 11. Will the spectral lines $5890 \text{ \AA}$ (2nd order) and $4000 \text{ \AA}$ (3rd order) overlap in a grating spectrum?

Overlap requires $n\lambda$ equal: $2 \times 5890 = 11780 \text{ \AA}$; $3 \times 4000 = 12000 \text{ \AA}$. These are close but not equal.

For visible light (4000 – $7000 \text{ \AA}$), the 2nd order spans 8000 – $14000 \text{ \AA}$ (in $n\lambda$) and the 3rd order spans 12000 – $21000 \text{ \AA}$, so the red end of the 2nd order overlaps the violet end of the 3rd order.

Answer: These two lines do not coincide exactly, but 2nd- and 3rd-order spectra of visible light partly overlap.



Problem 12. A grating of width 2 cm has 12000 lines. Can it resolve the lines 5890 Å and 5896 Å in the first order?

Required $R = \lambda/d\lambda \approx 5893/6 \approx 982$. Available $R = nN = 1 \times 12000 = 12000$.

Answer: Yes, easily (12000 > 982).

2.18 Practice problems with solutions: Module II

Q1. Two points on a wave of wavelength 2 m are 0.5 m apart. Find the phase difference between them.

$$\Delta\phi = (2\pi/\lambda)\Delta x = (2\pi/2)(0.5) = \pi/2.$$

Answer: $\pi/2$ (90°).

Q2. A wave of frequency 500 Hz travels at 350 m/s. How far apart are two points 60° out of phase?

$$\lambda = 350/500 = 0.7 \text{ m. } \Delta x = (\lambda/2\pi)(\pi/3) = \lambda/6.$$

Answer: 0.117 m.

Q3. Two coherent waves of amplitudes 3 and 4 units superpose with phase difference 90°. Find the resultant amplitude.

$$R^2 = 9 + 16 + 0 = 25.$$

Answer: $R = 5$ units.

Q4. A thin film ($\mu = 1.5$) of thickness 2×10^{-5} cm is illuminated normally by white light. Which visible wavelengths are missing in reflected light?

$$\text{Dark in reflection: } 2\mu t = n\lambda \Rightarrow \lambda = 2 \times 1.5 \times 2 \times 10^{-7}/n = 6 \times 10^{-7}/n \text{ m.}$$

$$n = 1: 6000 \text{ Å (visible). } n = 2: 3000 \text{ Å (UV).}$$

Answer: 6000 Å is missing.

Q5. Find the smallest thickness of a soap film ($\mu = 1.33$) that appears bright in reflected light of wavelength 6000 Å at normal incidence.

$$2\mu t = \lambda/2 \Rightarrow t = \lambda/(4\mu) = 6000/5.32.$$

Answer: $t \approx 1128 \text{ Å}$.

Q6. Newton's rings are observed with $\lambda = 6000 \text{ Å}$ and $R = 1 \text{ m}$. Find the diameter of the 5th dark ring.

$$D^2 = 4n\lambda R = 4 \times 5 \times 6 \times 10^{-7} \times 1 = 1.2 \times 10^{-5} \text{ m}^2.$$

Answer: $D \approx 3.46 \text{ mm}$.

Q7. In Newton's rings, the diameter of the 3rd dark ring is 0.3 cm. Find the diameter of the 12th dark ring.

$$D \propto \sqrt{n} \Rightarrow D_{12} = D_3 \sqrt{(12/3)} = 0.3 \times 2.$$

Answer: 0.6 cm.



Q8. Why is the centre of Newton's rings dark in reflected light?

At the point of contact the film thickness is zero, so the geometrical path difference is 0. The ray reflected from the lower (glass) surface suffers a phase change of π , so the effective path difference is $\lambda/2$, the condition for destructive interference.

Answer: Because of the extra $\lambda/2$ path due to reflection at the denser medium.

Q9. A slit of width 0.12 mm is illuminated by light of wavelength 6000 Å. Find the angular half-width of the central maximum.

$$\sin \theta = \lambda/a = 6 \times 10^{-7}/1.2 \times 10^{-4} = 5 \times 10^{-3}.$$

Answer: $\theta \approx 5 \times 10^{-3}$ rad $\approx 0.29^\circ$.

Q10. A grating with 6000 lines/cm is used with light of 5000 Å. How many orders are visible?

$$(a + b) = 1/6000 \text{ cm} = 1.667 \times 10^{-6} \text{ m. } n_{\text{max}} = 1.667 \times 10^{-6}/5 \times 10^{-7} = 3.33.$$

Answer: 3 orders on each side.

Q11. In a grating, the opaque spaces are twice the width of the transparent spaces ($b = 2a$). Which orders are absent?

$$\text{Absent orders: } n = m(a + b)/a = 3m.$$

Answer: 3rd, 6th, 9th, ... orders are missing.

2.19 MCQs: Module II

1. The relation between wave speed, frequency and wavelength is

(a) $v = \nu/\lambda$ (b) $v = \nu + \lambda$ (c) $v = \nu\lambda$ (d) $v = \lambda/\nu$

□ (c) — Definition.

2. The wave number k equals

(a) $2\pi\lambda$ (b) $2\pi/\lambda$ (c) $1/\lambda^2$ (d) $\lambda/2\pi$

□ (b) — $k = 2\pi/\lambda$.

3. The one-dimensional wave equation is

(a) $\partial^2 y/\partial t^2 = v^2 \partial^2 y/\partial x^2$ (b) $\partial^2 y/\partial x^2 = v \partial y/\partial t$ (c) $\partial y/\partial t = v^2 \partial^2 y/\partial x^2$ (d) $\partial y/\partial t = v \partial y/\partial x$

□ (a) — Second order in x and t .

4. The speed of a transverse wave on a string is

(a) μT (b) $\sqrt{(\mu/T)}$ (c) $\sqrt{(T/\mu)}$ (d) T/μ

□ (c) — T tension, μ mass per length.

5. Phase difference corresponding to a path difference of $\lambda/2$ is

(a) $\pi/4$ (b) π (c) $\pi/2$ (d) 2π

□ (b) — $(2\pi/\lambda)(\lambda/2)$.

6. For constructive interference, the path difference must be

(a) $(2n + 1)\lambda/2$ (b) $(n + 1/4)\lambda$ (c) $n\lambda$ (d) $\lambda/4$

□ (c) — Waves in phase.

7. Two waves of equal amplitude meet in phase. The resultant intensity is proportional to

(a) $4a^2$ (b) 0 (c) a^2 (d) $2a^2$

□ (a) — $R = 2a$.



- 8. Coherent sources have**
(a) the same amplitude only (b) constant phase difference and same frequency (c) different frequencies (d) random phases
□ (b) — Definition.
- 9. Two independent sodium lamps cannot produce interference because they are**
(a) too bright (b) not coherent (c) too far apart (d) of different colours
□ (b) — Random phases.
- 10. Young's double slit uses**
(a) division of amplitude (b) diffraction only (c) polarisation (d) division of wavefront
□ (d) — Wavefront split by slits.
- 11. Newton's rings use**
(a) refraction only (b) division of amplitude (c) total internal reflection (d) division of wavefront
□ (b) — Partial reflection at air film.
- 12. The effective path difference for reflection from a thin film is**
(a) $2\mu t \cos r - \lambda/2$ (b) $2\mu t \cos r$ (c) μt (d) $2t$
□ (a) — Extra $\lambda/2$ from reflection at denser medium.
- 13. The condition for a dark fringe in reflected light from a thin film is**
(a) $t = n\lambda$ (b) $2\mu t \cos r = (2n - 1)\lambda/2$ (c) $\mu t = \lambda$ (d) $2\mu t \cos r = n\lambda$
□ (d) — Accounts for the $\lambda/2$ shift.
- 14. An extremely thin film viewed in reflected light appears**
(a) coloured (b) white (c) dark (d) bright
□ (c) — Path difference only $\lambda/2$.
- 15. The fringes in reflected and transmitted light from a thin film are**
(a) both dark (b) both bright (c) complementary (d) identical
□ (c) — Energy conservation.
- 16. Colours in soap bubbles are due to**
(a) diffraction (b) polarisation (c) interference (d) dispersion
□ (c) — Thin-film interference.
- 17. The thickness of a quarter-wave anti-reflection coating is**
(a) $\lambda\mu$ (b) $\lambda/2\mu$ (c) $\lambda/4\mu$ (d) $4\lambda\mu$
□ (c) — $\mu t = \lambda/4$.
- 18. The fringe width in an air wedge of angle θ is**
(a) λ/θ (b) $2\lambda\theta$ (c) $\lambda/2\theta$ (d) $\lambda\theta$
□ (c) — $\beta = \lambda/2\mu\theta$ with $\mu = 1$.
- 19. In Newton's rings, the centre is dark in reflected light because**
(a) of phase change of π on reflection (b) of absorption (c) of diffraction (d) the lens is opaque
□ (a) — $t = 0$ but extra $\lambda/2$.
- 20. The diameter of the n-th dark ring in Newton's rings is proportional to**
(a) $\sqrt{n}$ (b) n^2 (c) $1/n$ (d) n
□ (a) — $D^2 = 4n\lambda R$.
- 21. In Newton's rings, the rings are**
(a) equally spaced (b) straight lines (c) farther apart as n increases (d) closer together as n increases
□ (d) — $D \propto \sqrt{n}$.



- 22.** The wavelength in Newton's rings experiment is given by
(a) $(D_{n+p}^2 - D_n^2)/4pR$ (b) $D^2/4R$ (c) $4pR/(D_{n+p}^2 - D_n^2)$ (d) $(D_{n+p}^2 - D_n^2)/4pR$
□ (d) — Standard formula.
- 23.** When water is introduced between lens and plate in Newton's rings, the ring diameters
(a) become zero (b) decrease (c) remain same (d) increase
□ (b) — $D^2 \propto 1/\mu$.
- 24.** In transmitted light, the central spot of Newton's rings is
(a) absent (b) bright (c) coloured (d) dark
□ (b) — Complementary pattern.
- 25.** The fringes in Newton's rings are circles because they are fringes of
(a) equal width (b) equal inclination (c) equal thickness (d) diffraction
□ (c) — Thickness constant on circles.
- 26.** Diffraction is significant when the aperture size is
(a) much larger than λ (b) infinite (c) comparable to λ (d) zero
□ (c) — Bending around edges.
- 27.** According to Huygens' principle, each point on a wavefront acts as
(a) a mirror (b) a source of secondary wavelets (c) an absorber (d) a lens
□ (b) — Statement.
- 28.** In Fraunhofer diffraction, the incident wavefront is
(a) elliptical (b) plane (c) spherical (d) cylindrical
□ (b) — Source at infinity.
- 29.** In Fresnel diffraction, the source and screen are at
(a) finite distances (b) the aperture (c) infinite distance (d) the same point
□ (a) — No lenses required.
- 30.** The condition for minima in single-slit diffraction is
(a) $a \sin \theta = (2n + 1)\lambda/2$ (b) $a \sin \theta = n\lambda$ (c) $a = n\lambda$ (d) $a \cos \theta = n\lambda$
□ (b) — Half-zone pairing.
- 31.** In single-slit diffraction, if the slit width is decreased, the central maximum
(a) becomes wider (b) does not change (c) becomes narrower (d) disappears
□ (a) — Width $\propto 1/a$.
- 32.** The intensity in single-slit diffraction is given by
(a) I_0/β (b) $I_0(\sin \beta/\beta)^2$ (c) $I_0 \cos^2 \beta$ (d) $I_0 \beta^2$
□ (b) — $\beta = \pi a \sin \theta/\lambda$.
- 33.** The intensity of the first secondary maximum in single-slit diffraction is about
(a) 1% (b) 25% (c) 50% (d) 4.5%
□ (d) — $\approx I_0/22$.
- 34.** The grating equation is
(a) $a \sin \theta = n\lambda$ (b) $(a + b) \sin \theta = n\lambda$ (c) $(a + b) \cos \theta = n\lambda$ (d) $(a - b) \sin \theta = n\lambda$
□ (b) — Principal maxima.
- 35.** A grating has 5000 lines/cm. Its grating element is
(a) 2×10^{-4} m (b) 5×10^{-5} m (c) 5000 cm (d) 2×10^{-4} cm
□ (d) — $1/N$.
- 36.** The maximum order visible with a grating is
(a) $(a + b)/\lambda$ (integer part) (b) 1 (c) $\lambda/(a + b)$ (d) $(a + b)\lambda$
□ (a) — $\sin \theta \leq 1$.



- 37.** In the grating spectrum of white light, the colour deviated most is
(a) violet (b) blue (c) red (d) green
□ (c) — Longest wavelength, largest θ .
- 38.** The resolving power of a grating is
(a) n/N (b) N/n (c) nN (d) λ/N
□ (c) — $R = \lambda/d\lambda = nN$.
- 39.** The zero-order image in a grating spectrum of white light is
(a) red (b) white (c) dark (d) violet
□ (b) — All wavelengths at $\theta = 0$.
- 40.** Diffraction fringes differ from interference fringes because they are
(a) perfectly dark at minima always (b) of unequal width and decreasing intensity (c) equally spaced (d) all equally bright
□ (b) — Comparison table.

Module III: Electromagnetism

Maxwell's four equations describe every electric and magnetic phenomenon, from a charged comb to radio transmission. They also predict that light is an electromagnetic wave. To write them compactly we first need the language of vector calculus.

3.1 Scalar and vector fields

- A **scalar field** assigns a number to every point in space: temperature $T(x, y, z)$, electric potential $V(x, y, z)$, pressure.
- A **vector field** assigns a vector to every point: electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, velocity of a fluid $\mathbf{v}$.
- The **del operator** (nabla) is the vector differential operator $\nabla = \mathbf{i} \partial/\partial x + \mathbf{j} \partial/\partial y + \mathbf{k} \partial/\partial z$. It acts on fields in three ways: gradient, divergence and curl.

3.2 Gradient of a scalar field

$$\text{grad } \phi = \nabla\phi = \mathbf{i} \partial\phi/\partial x + \mathbf{j} \partial\phi/\partial y + \mathbf{k} \partial\phi/\partial z$$

- $\nabla\phi$ is a **vector**. Its direction is the direction in which ϕ increases most rapidly, and its magnitude is that maximum rate of increase.
- $\nabla\phi$ is perpendicular (normal) to the level surface $\phi = \text{constant}$ at every point.
- **Directional derivative** of ϕ in the direction of a unit vector $\hat{\mathbf{n}}$ is $\nabla\phi \cdot \hat{\mathbf{n}}$.
- **Physical example:** the electric field is the negative gradient of the potential: $\mathbf{E} = -\nabla V$. Heat flows along $-\nabla T$.

3.3 Divergence of a vector field

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \partial A_x/\partial x + \partial A_y/\partial y + \partial A_z/\partial z$$

- $\nabla \cdot \mathbf{A}$ is a **scalar**. It measures the net outward flux of $\mathbf{A}$ per unit volume at a point, i.e. how much the field "spreads out" (source) or "converges" (sink) there.
- $\nabla \cdot \mathbf{A} > 0$: source (e.g. positive charge for $\mathbf{E}$). $\nabla \cdot \mathbf{A} < 0$: sink. $\nabla \cdot \mathbf{A} = 0$: **solenoidal** field (no sources or sinks, e.g. magnetic field $\mathbf{B}$, velocity of an incompressible fluid).
- Example: for $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\nabla \cdot \mathbf{r} = 3$.

3.4 Curl of a vector field

$$\begin{aligned} \text{curl } \mathbf{A} = \nabla \times \mathbf{A} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} \\ &= \mathbf{i}(\partial A_z/\partial y - \partial A_y/\partial z) + \mathbf{j}(\partial A_x/\partial z - \partial A_z/\partial x) + \mathbf{k}(\partial A_y/\partial x - \partial A_x/\partial y) \end{aligned}$$

- $\nabla \times \mathbf{A}$ is a **vector**. It measures the tendency of the field to rotate (circulate) about a point. A small paddle wheel placed in a fluid would spin about the direction of $\nabla \times \mathbf{v}$.
- If $\nabla \times \mathbf{A} = 0$ everywhere, $\mathbf{A}$ is **irrotational** (conservative). Then $\mathbf{A} = \nabla\phi$ for some scalar potential ϕ . The electrostatic field is irrotational: $\nabla \times \mathbf{E} = 0$.
- Example: for $\mathbf{v} = \omega \times \mathbf{r}$ (rigid rotation), $\nabla \times \mathbf{v} = 2\omega$.

Useful identities.

$$\begin{aligned} \nabla \times (\nabla\phi) &= 0 \text{ (curl of a gradient is zero)} \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0 \text{ (divergence of a curl is zero)} \\ \nabla \cdot (\nabla\phi) &= \nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 \text{ (Laplacian)} \\ \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \end{aligned}$$

3.5 Gauss divergence theorem (statement)

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{A}) dV$$

The total outward flux of a vector field through a closed surface S equals the volume integral of its divergence over the region V enclosed by S . It converts a surface integral into a volume integral (and vice versa).

3.6 Stokes' theorem (statement)

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

The line integral (circulation) of a vector field around a closed curve C equals the surface integral of its curl over any surface S bounded by C . It converts a line integral into a surface integral.

Note: These two theorems are the bridge between the integral form and the differential form of Maxwell's equations. Remember: Gauss relates flux to divergence; Stokes relates circulation to curl.

3.7 Four fundamental laws (integral form)

(1) Gauss's law for electricity. The total electric flux through a closed surface equals $1/\epsilon_0$ times the charge enclosed: $\oint \mathbf{E} \cdot d\mathbf{S} = q_{\text{enc}}/\epsilon_0 = (1/\epsilon_0) \iiint \rho dV$.

(2) Gauss's law for magnetism. Magnetic monopoles do not exist, so the net magnetic flux through any closed surface is zero: $\oint \mathbf{B} \cdot d\mathbf{S} = 0$.

(3) Faraday's law of electromagnetic induction. The emf around a closed loop equals the negative rate of change of magnetic flux through it: $\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi_B/dt = -\iint (\partial \mathbf{B}/\partial t) \cdot d\mathbf{S}$.

(4) Ampère's circuital law (with Maxwell's correction). $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0(I + I_d)$, where $I_d = \epsilon_0 d\Phi_E/dt$ is the displacement current.

3.8 Displacement current

Ampère's original law $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$ fails for a charging capacitor: for a surface passing between the plates, no conduction current crosses it, yet a magnetic field exists. Taking the divergence of $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ gives $\nabla \cdot \mathbf{J} = 0$, which contradicts the continuity equation $\nabla \cdot \mathbf{J} = -\partial \rho/\partial t$ for time-varying charges.

Maxwell added the **displacement current density** $\mathbf{J}_d = \epsilon_0 \partial \mathbf{E}/\partial t$. It is produced by a changing electric field, not by moving charges. Then $\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \partial \mathbf{E}/\partial t)$, which is consistent with charge conservation. Between capacitor plates, $I_d = \epsilon_0 A dE/dt = C dV/dt$ = conduction current in the wires.

3.9 Derivation of Maxwell's equations in differential form

(1) From Gauss's law for electricity. $\oint \mathbf{E} \cdot d\mathbf{S} = (1/\epsilon_0) \iiint \rho dV$. By the divergence theorem, the left side = $\iiint (\nabla \cdot \mathbf{E}) dV$. Since the result holds for every volume, the integrands must be equal: $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$.

(2) From Gauss's law for magnetism. $\oint \mathbf{B} \cdot d\mathbf{S} = 0 \Rightarrow \iiint (\nabla \cdot \mathbf{B}) dV = 0$ for every volume $\Rightarrow \nabla \cdot \mathbf{B} = 0$.

(3) From Faraday's law. $\oint \mathbf{E} \cdot d\mathbf{l} = -\iint (\partial \mathbf{B}/\partial t) \cdot d\mathbf{S}$. By Stokes' theorem, the left side = $\iint (\nabla \times \mathbf{E}) \cdot d\mathbf{S}$. Since this holds for every surface, $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$.

(4) From the Ampère-Maxwell law. $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \iint (\mathbf{J} + \epsilon_0 \partial \mathbf{E} / \partial t) \cdot d\mathbf{S}$. By Stokes' theorem, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$.

Summary: Maxwell's equations.

Law	Differential form	Integral form
Gauss (electric)	$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$	$\oint \mathbf{E} \cdot d\mathbf{S} = q / \epsilon_0$
Gauss (magnetic)	$\nabla \cdot \mathbf{B} = 0$	$\oint \mathbf{B} \cdot d\mathbf{S} = 0$
Faraday	$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$	$\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi_B / dt$
Ampère-Maxwell	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$	$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I + \mu_0 \epsilon_0 d\Phi_E / dt$

In a material medium, use $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{H} = \mathbf{B} / \mu$: $\nabla \cdot \mathbf{D} = \rho$, $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$, $\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t$.

Physical meaning. (1) Electric charges are sources of $\mathbf{E}$. (2) There are no magnetic charges; $\mathbf{B}$ lines form closed loops. (3) A changing magnetic field produces a circulating electric field (the basis of generators and transformers). (4) Currents and changing electric fields produce circulating magnetic fields.

3.10 Electromagnetic wave equation in vacuum

In free space there are no charges or currents: $\rho = 0$, $\mathbf{J} = 0$. Maxwell's equations become $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$, $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$.

Take the curl of Faraday's law: $\nabla \times (\nabla \times \mathbf{E}) = -\partial (\nabla \times \mathbf{B}) / \partial t = -\mu_0 \epsilon_0 \partial^2 \mathbf{E} / \partial t^2$.

Using the identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ and $\nabla \cdot \mathbf{E} = 0$:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \partial^2 \mathbf{E} / \partial t^2 \text{ and similarly } \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \partial^2 \mathbf{B} / \partial t^2$$

Comparing with the wave equation $\nabla^2 \psi = (1/v^2) \partial^2 \psi / \partial t^2$, the fields travel as waves with speed

$$c = 1/\sqrt{\mu_0 \epsilon_0} = 1/\sqrt{(4\pi \times 10^{-7} \times 8.854 \times 10^{-12})} \approx 3 \times 10^8 \text{ m/s}$$

This is the measured speed of light, so Maxwell concluded that **light is an electromagnetic wave**.

In a medium of permeability $\mu = \mu_r \mu_0$ and permittivity $\epsilon = \epsilon_r \epsilon_0$: $v = 1/\sqrt{\mu \epsilon} = c/\sqrt{(\mu_r \epsilon_r)}$, and the refractive index is $n = c/v = \sqrt{(\mu_r \epsilon_r)}$.

3.11 Transverse nature of electromagnetic waves

Take a plane wave travelling along z : $\mathbf{E} = \mathbf{E}_0 e^{i(kz - \omega t)}$.

- From $\nabla \cdot \mathbf{E} = 0$: $\partial E_z / \partial z = ikE_z = 0 \Rightarrow E_z = 0$. So $\mathbf{E}$ has no component along the direction of propagation.
- Similarly $\nabla \cdot \mathbf{B} = 0 \Rightarrow B_z = 0$.
- From $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$: $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$, so $\mathbf{B}$ is perpendicular to both $\mathbf{k}$ and $\mathbf{E}$.

$$\mathbf{E} \perp \mathbf{B} \perp \text{direction of propagation; } |\mathbf{E}|/|\mathbf{B}| = c; \mathbf{E} \text{ and } \mathbf{B} \text{ oscillate in phase}$$

Hence EM waves are **transverse**. The direction of propagation is along $\mathbf{E} \times \mathbf{B}$. Energy flow per unit area per second is given by the **Poynting vector** $\mathbf{S} = (1/\mu_0) \mathbf{E} \times \mathbf{B}$. The average intensity of a plane wave is $I = (1/2) c \epsilon_0 E_0^2$. The energy is shared equally between the electric and magnetic fields: $(1/2) \epsilon_0 E^2 = B^2 / (2\mu_0)$.

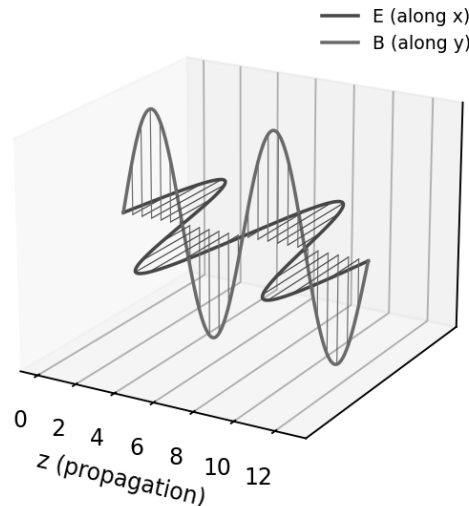


Figure 3.1: A plane electromagnetic wave. E and B are perpendicular to each other and to the direction of propagation, and oscillate in phase.

3.12 Electromagnetic waves in a conducting medium

In a conductor of conductivity σ , $\mathbf{J} = \sigma \mathbf{E}$ and (for a good conductor) $\rho = 0$ inside.

Maxwell's fourth equation becomes $\nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \mu\epsilon \partial\mathbf{E}/\partial t$. Taking the curl of Faraday's law as before:

$$\nabla^2 \mathbf{E} = \mu\sigma \partial\mathbf{E}/\partial t + \mu\epsilon \partial^2 \mathbf{E}/\partial t^2$$

The first term (conduction current) causes damping. For a plane wave $\mathbf{E} = E_0 e^{i(kz - \omega t)}$, the wave number becomes complex: $k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$, i.e. $k = \alpha + i\beta$. The wave is

$$\mathbf{E} = E_0 e^{-\beta z} e^{i(\alpha z - \omega t)}$$

The amplitude decreases exponentially with depth: the wave is **attenuated** as its energy is converted into Joule heat.

Good conductor ($\sigma \gg \omega\epsilon$). Then $k^2 \approx i\mu\sigma\omega$, so $\alpha = \beta = \sqrt{(\mu\sigma\omega/2)}$.

$$\text{Skin depth } \delta = 1/\beta = \sqrt{2/(\mu\sigma\omega)}$$

The **skin depth** is the distance in which the amplitude falls to $1/e$ (37%) of its surface value. At high frequencies δ is very small, so AC currents and EM fields are confined to a thin surface layer (**skin effect**). For copper ($\sigma = 5.8 \times 10^7 \text{ S/m}$) at 60 Hz, $\delta \approx 8.5 \text{ mm}$; at 1 MHz, $\delta \approx 0.066 \text{ mm}$; at 10 GHz, $\delta \approx 0.66 \text{ }\mu\text{m}$.

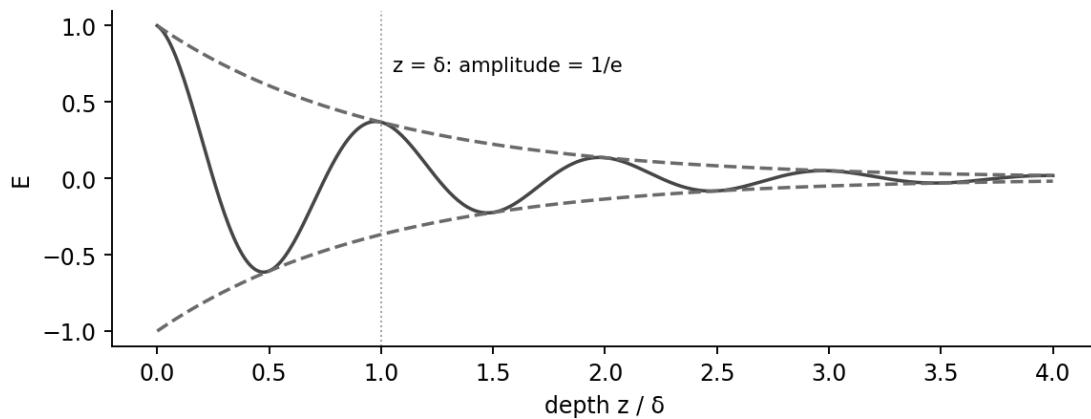


Figure 3.2: Attenuation of an EM wave inside a conductor. The amplitude falls to $1/e$ at the skin depth δ .

Consequences. Metals reflect light and radio waves (mirrors, shielding); submarines communicate with very low frequency waves; high-frequency conductors are made hollow or silver-plated; microwave ovens have metal walls.

3.13 Common mistakes to avoid in Module III

- Treating $\nabla \cdot \mathbf{A}$ as a vector or $\nabla \times \mathbf{A}$ as a scalar. Divergence is a scalar; curl is a vector.
- Forgetting the minus sign in Faraday's law (Lenz's law).
- Leaving out the displacement current in Ampère's law; without it there are no EM waves.
- Writing $c = \sqrt{(\mu_0 \epsilon_0)}$ instead of $1/\sqrt{(\mu_0 \epsilon_0)}$.
- Stating that $\mathbf{B}$ is parallel to $\mathbf{E}$ in an EM wave. They are perpendicular and in phase.

3.14 Physical meaning of divergence and curl (derivation)

Divergence as flux per unit volume. Consider a small box of sides dx , dy , dz at (x, y, z) . The flux of $\mathbf{A}$ out through the face at $x + dx$ is $A_x(x + dx)dy dz$ and in through the face at x is $A_x(x)dy dz$. Net outward flux through these two faces = $[A_x(x + dx) - A_x(x)]dy dz = (\partial A_x / \partial x)dx dy dz$. Adding the other two pairs of faces, net outward flux = $(\partial A_x / \partial x + \partial A_y / \partial y + \partial A_z / \partial z)dV = (\nabla \cdot \mathbf{A})dV$. So $\nabla \cdot \mathbf{A} = \lim (\text{net outward flux})/(\text{volume})$. This is also the proof idea of Gauss's divergence theorem: adding the flux from all tiny boxes, interior faces cancel, leaving the flux through the outer surface.

Curl as circulation per unit area. Take a small rectangle $dx \times dy$ in the xy -plane. The line integral of $\mathbf{A}$ around it is $[A_x(y) - A_x(y + dy)]dx + [A_y(x + dx) - A_y(x)]dy = (\partial A_y / \partial x - \partial A_x / \partial y)dx dy$. So $(\nabla \times \mathbf{A})_z = \lim (\text{circulation})/(\text{area})$. Adding the contributions of many small loops gives Stokes' theorem (inner edges cancel).

3.15 Equation of continuity

Charge is conserved. The rate at which charge leaves a volume through its surface equals the rate of decrease of charge inside: $\oint \mathbf{J} \cdot d\mathbf{S} = -d/dt \iiint \rho dV$. Using the divergence theorem:

$$\nabla \cdot \mathbf{J} + \partial \rho / \partial t = 0 \text{ (equation of continuity)}$$

For steady currents ($\partial \rho / \partial t = 0$), $\nabla \cdot \mathbf{J} = 0$: current lines form closed loops (Kirchhoff's current law in differential form).

Inconsistency of Ampère's original law. Taking divergence of $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ gives $0 = \mu_0 \nabla \cdot \mathbf{J}$, i.e. $\nabla \cdot \mathbf{J} = 0$ always, which contradicts the continuity equation when ρ changes with time. With Maxwell's term: $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 [\nabla \cdot \mathbf{J} + \varepsilon_0 \partial(\nabla \cdot \mathbf{E})/\partial t] = \mu_0 [\nabla \cdot \mathbf{J} + \partial\rho/\partial t] = 0$ □.

3.16 Energy in electromagnetic fields and Poynting's theorem (brief)

Energy density of the electromagnetic field: $u = (1/2)\varepsilon_0 E^2 + B^2/(2\mu_0)$. From Maxwell's equations one can show

$$-\partial u/\partial t = \nabla \cdot \mathbf{S} + \mathbf{J} \cdot \mathbf{E}, \quad \mathbf{S} = (1/\mu_0) \mathbf{E} \times \mathbf{B}$$

The decrease of field energy in a region equals the energy flowing out through its surface (flux of $\mathbf{S}$) plus the work done on charges (Joule heating $\mathbf{J} \cdot \mathbf{E}$). $\mathbf{S}$ is the Poynting vector (W/m^2). For a plane wave, $|\mathbf{S}| = c\varepsilon_0 E^2$, and its time average is $I = (1/2)c\varepsilon_0 E_0^2$.

Radiation pressure. EM waves carry momentum $p = U/c$. A wave of intensity I falling normally on a perfectly absorbing surface exerts pressure I/c ; on a perfect reflector, $2I/c$. This is why sunlight exerts a tiny pressure used in solar sails.

3.17 Wave equation for $\mathbf{B}$ and the plane-wave solution

Taking the curl of the Ampère–Maxwell law in vacuum: $\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \partial(\nabla \times \mathbf{E})/\partial t = -\mu_0 \varepsilon_0 \partial^2 \mathbf{B}/\partial t^2$. Using $\nabla \cdot \mathbf{B} = 0$, $\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \partial^2 \mathbf{B}/\partial t^2$.

Plane-wave solution. $\mathbf{E} = \mathbf{E}_0 \cos(kz - \omega t)$, $\mathbf{B} = \mathbf{B}_0 \cos(kz - \omega t)$ with $\omega/k = c$.

Substituting in Faraday's law ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$) for $\mathbf{E} = E_0 \mathbf{i} \cos(kz - \omega t)$: $\nabla \times \mathbf{E} = -kE_0 \mathbf{j} \sin(kz - \omega t)$ and $-\partial \mathbf{B}/\partial t = -\omega B_0 \mathbf{j} \sin(kz - \omega t)$ (direction $\mathbf{j}$). So $\mathbf{B}$ is along $\mathbf{j}$ and $B_0 = kE_0/\omega = E_0/c$. Hence $\mathbf{E} \perp \mathbf{B} \perp \mathbf{k}$ and $E = cB$.

Impedance of free space. $E/H = \mu_0 c = \sqrt{(\mu_0/\varepsilon_0)} \approx 377 \, \Omega$.

3.18 Wave propagation in a conducting medium: detailed solution

Start with $\nabla^2 \mathbf{E} = \mu\sigma \partial \mathbf{E}/\partial t + \mu\varepsilon \partial^2 \mathbf{E}/\partial t^2$. Try $\mathbf{E} = \mathbf{E}_0 e^{i(kz - \omega t)}$: $-k^2 = -i\mu\sigma\omega - \mu\varepsilon\omega^2 \Rightarrow k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$.

Good conductor ($\sigma/\omega\varepsilon \gg 1$). $k^2 \approx i\mu\sigma\omega$, $k = \sqrt{(\mu\sigma\omega)} e^{i\pi/4} = \sqrt{(\mu\sigma\omega/2)}(1 + i)$. Write $k = \alpha + i\beta$ with $\alpha = \beta = \sqrt{(\mu\sigma\omega/2)} = 1/\delta$.

Then $\mathbf{E} = \mathbf{E}_0 e^{-z/\delta} e^{i(z/\delta - \omega t)}$.

- The amplitude decays as $e^{-z/\delta}$.
- The wavelength inside the conductor is $\lambda = 2\pi/\alpha = 2\pi\delta$ (very short).
- The magnetic field lags the electric field by 45° , and B/E is much larger than in vacuum.

Poor conductor / dielectric ($\sigma/\omega\varepsilon \ll 1$). The wave propagates with speed $1/\sqrt{(\mu\varepsilon)}$ and weak attenuation; the attenuation constant is $\beta \approx (\sigma/2)\sqrt{(\mu/\varepsilon)}$, independent of frequency.

Quantity	Vacuum / dielectric	Good conductor
Wave number	$k = \omega\sqrt{(\mu\varepsilon)}$, real	$k = (1 + i)/\delta$, complex
Amplitude	Constant	Decays as $e^{-z/\delta}$
Phase of $\mathbf{B}$ relative to $\mathbf{E}$	In phase	$\mathbf{B}$ lags $\mathbf{E}$ by 45°
Penetration	Unlimited	About $\delta = \sqrt{(2/\mu\sigma\omega)}$

3.19 Solved problems: Module III



Problem 1. Find $\nabla\phi$ for $\phi = x^2y + y^2z + z^2x$ at the point $(1, -1, 2)$.

$$\partial\phi/\partial x = 2xy + z^2, \partial\phi/\partial y = x^2 + 2yz, \partial\phi/\partial z = y^2 + 2zx.$$

$$\text{At } (1, -1, 2): 2(1)(-1) + 4 = 2; 1 + 2(-1)(2) = -3; 1 + 2(2)(1) = 5.$$

Answer: $\nabla\phi = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$; $|\nabla\phi| = \sqrt{38}$.

Problem 2. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

$$\nabla\phi = (2xyz + 4z^2)\mathbf{i} + x^2z\mathbf{j} + (x^2y + 8xz)\mathbf{k}. \text{ At } (1, -2, -1): (4 + 4, -1, -2 - 8) = (8, -1, -10).$$

$$\text{Unit vector } \hat{n} = (2, -1, -2)/3.$$

$$\nabla\phi \cdot \hat{n} = (16 + 1 + 20)/3.$$

Answer: $37/3$.

Problem 3. Find the unit normal to the surface $x^2 + y^2 - z = 1$ at $(1, 1, 1)$.

$$\phi = x^2 + y^2 - z. \nabla\phi = (2x, 2y, -1) = (2, 2, -1) \text{ at } (1, 1, 1). |\nabla\phi| = 3.$$

Answer: $\hat{n} = (2\mathbf{i} + 2\mathbf{j} - \mathbf{k})/3$.

Problem 4. If $\mathbf{A} = xy^2\mathbf{i} + 2x^2yz\mathbf{j} - 3yz^2\mathbf{k}$, find $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$ at $(1, -1, 1)$.

$$\nabla \cdot \mathbf{A} = y^2 + 2x^2z - 6yz = 1 + 2 + 6 = 9.$$

$$(\nabla \times \mathbf{A})_x = \partial(-3yz^2)/\partial y - \partial(2x^2yz)/\partial z = -3z^2 - 2x^2y = -3 + 2 = -1.$$

$$(\nabla \times \mathbf{A})_y = \partial(xy^2)/\partial z - \partial(-3yz^2)/\partial x = 0. (\nabla \times \mathbf{A})_z = \partial(2x^2yz)/\partial x - \partial(xy^2)/\partial y = 4xyz - 2xy = -4 + 2 = -2.$$

Answer: $\nabla \cdot \mathbf{A} = 9$; $\nabla \times \mathbf{A} = -\mathbf{i} - 2\mathbf{k}$.

Problem 5. Show that $\mathbf{A} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$ is irrotational and find ϕ such that $\mathbf{A} = \nabla\phi$.

$$(\nabla \times \mathbf{A})_x = \partial(xy)/\partial y - \partial(zx)/\partial z = x - x = 0; \text{ similarly the other components are 0. So } \mathbf{A} \text{ is irrotational.}$$

$$\partial\phi/\partial x = yz \Rightarrow \phi = xyz + f(y, z); \text{ checking } \partial\phi/\partial y = xz \text{ and } \partial\phi/\partial z = xy \text{ gives } f = \text{constant.}$$

Answer: $\phi = xyz + C$.

Problem 6. Show that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has $\nabla \cdot \mathbf{r} = 3$ and $\nabla \times \mathbf{r} = 0$.

$$\nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3. (\nabla \times \mathbf{r})_x = \partial z/\partial y - \partial y/\partial z = 0, \text{ and similarly for the others.}$$

Answer: Shown: $\mathbf{r}$ is irrotational, with divergence 3.

Problem 7. Find the value of a so that $\mathbf{A} = (x + 3y)\mathbf{i} + (y - 2z)\mathbf{j} + (x + az)\mathbf{k}$ is solenoidal.

$$\nabla \cdot \mathbf{A} = 1 + 1 + a = 0.$$

Answer: $a = -2$.

Problem 8. Verify Gauss's divergence theorem for $\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the cube $0 \leq x, y, z \leq 1$.



Volume integral: $\nabla \cdot \mathbf{A} = 3$, so $\iiint 3 \, dV = 3 \times 1 = 3$.

Surface integral: on the face $x = 1$, $\mathbf{A} \cdot \hat{n} = x = 1$, area 1 $\Rightarrow$ flux 1. On $x = 0$, $\mathbf{A} \cdot \hat{n} = -x = 0$. Similarly faces $y = 1$ and $z = 1$ contribute 1 each; the other three contribute 0. Total = 3.

Answer: Both sides equal 3. Verified.

Problem 9. Using Gauss's law, find the electric field at distance r from an infinitely long line charge of density λ .

Take a coaxial cylinder of radius r and length l as the Gaussian surface. By symmetry $\mathbf{E}$ is radial and constant on the curved surface; the ends contribute no flux.

$$E \times 2\pi r l = \lambda l / \epsilon_0.$$

Answer: $E = \lambda / (2\pi\epsilon_0 r)$.

Problem 10. Calculate the displacement current in a parallel-plate capacitor of capacitance $2 \, \mu\text{F}$ when the voltage changes at a rate of $10^6 \, \text{V/s}$.

$$I_d = \epsilon_0 A \, dE/dt = C \, dV/dt = 2 \times 10^{-6} \times 10^6.$$

Answer: $I_d = 2 \, \text{A}$.

Problem 11. Show that $c = 1/\sqrt{\mu_0\epsilon_0}$ and calculate its value.

From the wave equation derived from Maxwell's equations, $v^2 = 1/(\mu_0\epsilon_0)$.

$$\mu_0\epsilon_0 = 4\pi \times 10^{-7} \times 8.854 \times 10^{-12} = 1.1127 \times 10^{-17} \, \text{s}^2/\text{m}^2.$$

Answer: $c = 1/\sqrt{(1.1127 \times 10^{-17})} \approx 2.998 \times 10^8 \, \text{m/s}$.

Problem 12. The electric field amplitude of an EM wave in vacuum is $300 \, \text{V/m}$. Find the amplitude of the magnetic field and the average intensity.

$$B_0 = E_0/c = 300/(3 \times 10^8) = 10^{-6} \, \text{T}.$$

$$I = (1/2)c\epsilon_0 E_0^2 = 0.5 \times 3 \times 10^8 \times 8.854 \times 10^{-12} \times 9 \times 10^4 = 119.5 \, \text{W/m}^2.$$

Answer: $B_0 = 1 \, \mu\text{T}$; $I \approx 120 \, \text{W/m}^2$.

Problem 13. Sunlight has an intensity of about $1000 \, \text{W/m}^2$ at the Earth's surface. Find the peak electric and magnetic fields.

$$I = (1/2)c\epsilon_0 E_0^2 \Rightarrow E_0 = \sqrt{(2I/(c\epsilon_0))}. \text{ Here } c\epsilon_0 = 3 \times 10^8 \times 8.854 \times 10^{-12} = 2.656 \times 10^{-3}.$$

$$E_0 = \sqrt{(2000/2.656 \times 10^{-3})} = \sqrt{(7.53 \times 10^5)} = 868 \, \text{V/m}.$$

$$B_0 = E_0/c = 868/(3 \times 10^8) = 2.89 \times 10^{-6} \, \text{T}.$$

Answer: $E_0 \approx 868 \, \text{V/m}$; $B_0 \approx 2.9 \, \mu\text{T}$.

Problem 14. Find the refractive index of a non-magnetic medium of relative permittivity 2.25, and the speed of light in it.

$$n = \sqrt{(\mu_r \epsilon_r)} = \sqrt{2.25} = 1.5. \, v = c/n = 2 \times 10^8 \, \text{m/s}.$$

Answer: $n = 1.5$; $v = 2 \times 10^8 \, \text{m/s}$.

Problem 15. Find the skin depth in copper ($\sigma = 5.8 \times 10^7 \text{ S/m}$, $\mu \approx \mu_0$) at 1 MHz.

$$\delta = \sqrt{2/(\mu\sigma\omega)} = \sqrt{2/(4\pi \times 10^{-7} \times 5.8 \times 10^7 \times 2\pi \times 10^6)}.$$

$$\text{Denominator} = 1.2566 \times 10^{-6} \times 5.8 \times 10^7 \times 6.283 \times 10^6 = 4.579 \times 10^8. \delta = \sqrt{(4.368 \times 10^{-9})}.$$

Answer: $\delta \approx 6.6 \times 10^{-5} \text{ m} = 0.066 \text{ mm}$.

Problem 16. Find the skin depth of sea water ($\sigma = 4 \text{ S/m}$, $\mu = \mu_0$) for a 1 MHz radio wave. Why do submarines use very low frequencies?

$$\delta = \sqrt{2/(4\pi \times 10^{-7} \times 4 \times 2\pi \times 10^6)} = \sqrt{2/31.58} = 0.25 \text{ m}.$$

Since $\delta \propto 1/\sqrt{\omega}$, lower frequencies penetrate deeper (at 100 Hz, $\delta \approx 25 \text{ m}$).

Answer: $\delta \approx 0.25 \text{ m}$ at 1 MHz; very low frequencies are used because they penetrate much deeper into sea water.

3.20 More solved problems: Module III

Problem 1. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

$$\phi_1 = x^2 + y^2 + z^2 - 9: \nabla\phi_1 = (2x, 2y, 2z) = (4, -2, 4), |\nabla\phi_1| = 6.$$

$$\phi_2 = x^2 + y^2 - z - 3: \nabla\phi_2 = (2x, 2y, -1) = (4, -2, -1), |\nabla\phi_2| = \sqrt{21}.$$

$$\cos \theta = (16 + 4 - 4)/(6\sqrt{21}) = 16/(6 \times 4.583) = 0.582.$$

Answer: $\theta = \cos^{-1}(8/(3\sqrt{21})) \approx 54.4^\circ$.

Problem 2. Find $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ for $\mathbf{F} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j} + (y^2 - xy)\mathbf{k}$.

$$\nabla \cdot \mathbf{F} = 2x + 2x + 0 = 4x.$$

$$(\nabla \times \mathbf{F})_x = \partial(y^2 - xy)/\partial y - \partial(2xy)/\partial z = 2y - x. (\nabla \times \mathbf{F})_y = \partial(x^2 - y^2)/\partial z - \partial(y^2 - xy)/\partial x = 0 + y = y. (\nabla \times \mathbf{F})_z = \partial(2xy)/\partial x - \partial(x^2 - y^2)/\partial y = 2y + 2y = 4y.$$

Answer: $\nabla \cdot \mathbf{F} = 4x$; $\nabla \times \mathbf{F} = (2y - x)\mathbf{i} + y\mathbf{j} + 4y\mathbf{k}$.

Problem 3. Show that $\nabla^2(1/r) = 0$ for $r \neq 0$.

$$\nabla(1/r) = -\mathbf{r}/r^3. \nabla \cdot (\mathbf{r}/r^3) = (1/r^3)\nabla \cdot \mathbf{r} + \mathbf{r} \cdot \nabla(1/r^3) = 3/r^3 + \mathbf{r} \cdot (-3\mathbf{r}/r^5) = 3/r^3 - 3/r^3 = 0.$$

Answer: $\nabla^2(1/r) = -\nabla \cdot (\mathbf{r}/r^3) = 0$ (the Coulomb potential satisfies Laplace's equation away from the charge).

Problem 4. Evaluate $\oint \mathbf{A} \cdot d\mathbf{l}$ for $\mathbf{A} = -y\mathbf{i} + x\mathbf{j}$ around the circle $x^2 + y^2 = a^2$ using Stokes' theorem.

$$\nabla \times \mathbf{A} = (\partial x/\partial x - \partial(-y)/\partial y)\mathbf{k} = 2\mathbf{k}.$$

$$\iint (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = 2 \times (\text{area}) = 2\pi a^2.$$

Answer: $2\pi a^2$. (Direct: $x = a \cos t$, $y = a \sin t$ gives $\int_0^{2\pi} a^2 dt = 2\pi a^2$.)

Problem 5. Using the divergence theorem, evaluate $\oint \mathbf{r} \cdot d\mathbf{S}$ over a sphere of radius R .



$$\nabla \cdot \mathbf{r} = 3. \iiint 3 \, dV = 3 \times (4/3)\pi R^3.$$

Answer: $4\pi R^3$.

Problem 6. A point charge of $1 \, \mu\text{C}$ is at the centre of a cube of side $10 \, \text{cm}$. Find the electric flux through the cube and through one face.

$$\text{Total flux} = q/\epsilon_0 = 10^{-6}/8.854 \times 10^{-12} = 1.13 \times 10^5 \, \text{N m}^2/\text{C}.$$

By symmetry each of the six faces gets one-sixth.

Answer: Total $1.13 \times 10^5 \, \text{V m}$; each face $1.88 \times 10^4 \, \text{V m}$.

Problem 7. A parallel-plate capacitor with circular plates of radius $5 \, \text{cm}$ is being charged so that E between the plates changes at $10^{12} \, \text{V m}^{-1} \text{s}^{-1}$. Find the displacement current.

$$I_d = \epsilon_0 A \, dE/dt = 8.854 \times 10^{-12} \times \pi(0.05)^2 \times 10^{12} = 8.854 \times 7.854 \times 10^{-3}.$$

Answer: $I_d \approx 0.0695 \, \text{A} \approx 70 \, \text{mA}$.

Problem 8. The electric field of an EM wave in a medium is $E = 50 \sin(10^8 t - 0.5z) \, \text{V/m}$ (non-magnetic medium). Find the speed of the wave, refractive index and relative permittivity.

$$v = \omega/k = 10^8/0.5 = 2 \times 10^8 \, \text{m/s}. n = c/v = 1.5. \epsilon_r = n^2 = 2.25 (\mu_r = 1).$$

Answer: $v = 2 \times 10^8 \, \text{m/s}$; $n = 1.5$; $\epsilon_r = 2.25$.

Problem 9. The average intensity of a radio wave at a receiver is $10^{-9} \, \text{W/m}^2$. Find the rms electric field.

$$I = c\epsilon_0 E_{\text{rms}}^2 = E_{\text{rms}}^2 = \sqrt{(10^{-9}/(3 \times 10^8 \times 8.854 \times 10^{-12}))} = \sqrt{(3.77 \times 10^{-7})}.$$

Answer: $E_{\text{rms}} \approx 6.1 \times 10^{-4} \, \text{V/m}$.

Problem 10. Find the radiation pressure of sunlight ($1.4 \, \text{kW/m}^2$) on a perfectly reflecting surface.

$$P = 2I/c = 2 \times 1400/3 \times 10^8.$$

Answer: $P \approx 9.3 \times 10^{-6} \, \text{N/m}^2$.

Problem 11. Compare the conduction and displacement currents in sea water ($\sigma = 4 \, \text{S/m}$, $\epsilon_r = 81$) at $1 \, \text{MHz}$. Is sea water a good conductor at this frequency?

$$\text{Ratio } \sigma/(\omega\epsilon) = 4/(2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}) = 4/(4.506 \times 10^{-3}) \approx 888.$$

Answer: Conduction current is about 900 times larger, so sea water behaves as a good conductor at $1 \, \text{MHz}$.

Problem 12. At what frequency does sea water ($\sigma = 4 \, \text{S/m}$, $\epsilon_r = 81$) behave as a dielectric ($\sigma = \omega\epsilon$)?

$$\omega = \sigma/\epsilon = 4/(81 \times 8.854 \times 10^{-12}) = 5.58 \times 10^9 \, \text{rad/s}.$$

Answer: $\nu \approx 0.89$ GHz. Above about 1 GHz, sea water starts behaving like a lossy dielectric.

Problem 13. Find the skin depth of silver ($\sigma = 6.2 \times 10^7$ S/m) at 10 GHz.

$$\delta = \sqrt{2/(4\pi \times 10^{-7} \times 6.2 \times 10^7 \times 2\pi \times 10^{10})} = \sqrt{2/(4.896 \times 10^{12})} = \sqrt{4.085 \times 10^{-13}}.$$

Answer: $\delta \approx 0.64$ μ m. A silver plating of a few micrometres is enough for microwave components.

3.21 Practice problems with solutions: Module III

Q1. Find $\nabla\phi$ if $\phi = \log(x^2 + y^2 + z^2)$.

$$\partial\phi/\partial x = 2x/(x^2 + y^2 + z^2), \text{ and similarly for } y, z.$$

Answer: $\nabla\phi = 2\mathbf{r}/r^2$, where $r^2 = x^2 + y^2 + z^2$.

Q2. Show that $\nabla(1/r) = -\mathbf{r}/r^3$.

$$\partial(1/r)/\partial x = -(1/r^2)(\partial r/\partial x) = -(1/r^2)(x/r) = -x/r^3; \text{ similarly for } y, z.$$

Answer: $\nabla(1/r) = -\mathbf{r}/r^3$ (this is why the Coulomb field $\propto \mathbf{r}/r^3$).

Q3. Find $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$ for $\mathbf{A} = x^2y \mathbf{i} - 2xz \mathbf{j} + 2yz \mathbf{k}$.

$$\nabla \cdot \mathbf{A} = 2xy + 0 + 2y = 2y(x + 1).$$

$$\nabla \times \mathbf{A} = (\partial(2yz)/\partial y - \partial(-2xz)/\partial z)\mathbf{i} + (0 - 0)\mathbf{j} + (\partial(-2xz)/\partial x - \partial(x^2y)/\partial y)\mathbf{k} = (2z + 2x)\mathbf{i} + (-2z - x^2)\mathbf{k}.$$

Answer: $\nabla \cdot \mathbf{A} = 2y(x + 1)$; $\nabla \times \mathbf{A} = 2(x + z)\mathbf{i} - (x^2 + 2z)\mathbf{k}$.

Q4. Prove that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for any vector field $\mathbf{A}$.

$$\nabla \cdot (\nabla \times \mathbf{A}) = \partial/\partial x(\partial A_z/\partial y - \partial A_y/\partial z) + \partial/\partial y(\partial A_x/\partial z - \partial A_z/\partial x) + \partial/\partial z(\partial A_y/\partial x - \partial A_x/\partial y).$$

The mixed partial derivatives cancel in pairs (e.g. $\partial^2 A_z/\partial x \partial y - \partial^2 A_z/\partial y \partial x = 0$).

Answer: 0. (This is why $\nabla \cdot \mathbf{B} = 0$ is consistent with $\mathbf{B} = \nabla \times \mathbf{A}$.)

Q5. Using Gauss's law, find the field of a uniformly charged spherical shell (charge Q , radius R) outside and inside.

$$\text{Outside } (r > R): E \times 4\pi r^2 = Q/\epsilon_0 \Rightarrow E = Q/(4\pi\epsilon_0 r^2).$$

$$\text{Inside } (r < R): \text{the Gaussian sphere encloses no charge} \Rightarrow E = 0.$$

Answer: $E = Q/(4\pi\epsilon_0 r^2)$ outside; 0 inside.

Q6. Using Ampère's law, find B at distance r from a long straight wire carrying current I .

$$\text{Circular path of radius } r: B \times 2\pi r = \mu_0 I.$$

Answer: $B = \mu_0 I/(2\pi r)$.

Q7. The magnetic field of an EM wave is $B = 2 \times 10^{-8} \sin(\omega t - kz)$ T. Find the peak electric field.

$$E_0 = cB_0 = 3 \times 10^8 \times 2 \times 10^{-8}.$$

Answer: $E_0 = 6 \text{ V/m}$.

Q8. A plane EM wave has frequency 10 MHz in vacuum. Find its wavelength and wave number.

$$\lambda = c/\nu = 3 \times 10^8/10^7 = 30 \text{ m}; k = 2\pi/\lambda = 0.209 \text{ rad/m}.$$

Answer: $\lambda = 30 \text{ m}$; $k \approx 0.21 \text{ m}^{-1}$.

Q9. Show that the displacement current in a capacitor equals the conduction current in the connecting wires.

$$E = q/(\epsilon_0 A) \text{ between plates. } I_d = \epsilon_0 A \, dE/dt = \epsilon_0 A \times (1/\epsilon_0 A) \, dq/dt = dq/dt = I.$$

Answer: $I_d = I$ (continuity of total current).

Q10. At what frequency is the skin depth in aluminium ($\sigma = 3.5 \times 10^7 \text{ S/m}$) equal to 1 mm?

$$\delta^2 = 2/(\mu\sigma\omega) \Rightarrow \omega = 2/(\mu\sigma\delta^2) = 2/(4\pi \times 10^{-7} \times 3.5 \times 10^7 \times 10^{-6}) = 2/(4.398 \times 10^{-5}) \\ = 4.55 \times 10^4 \text{ rad/s}.$$

Answer: $\nu = \omega/2\pi \approx 7.2 \text{ kHz}$.

3.22 MCQs: Module III

1. The gradient of a scalar field is

- (a) a tensor (b) a vector (c) zero (d) a scalar

☐ (b) — $\nabla\phi$ is a vector.

2. The divergence of a vector field is

- (a) a tensor (b) a vector (c) a scalar (d) always zero

☐ (c) — $\nabla \cdot \mathbf{A}$ is scalar.

3. The curl of a vector field is

- (a) a number (b) always zero (c) a vector (d) a scalar

☐ (c) — $\nabla \times \mathbf{A}$ is vector.

4. $\nabla\phi$ is directed

- (a) normal to the level surface (b) at 45° to it (c) opposite to the surface (d) along the level surface

☐ (a) — Normal to $\phi = \text{constant}$.

5. A vector field with zero divergence is called

- (a) solenoidal (b) irrotational (c) conservative (d) harmonic

☐ (a) — No sources or sinks.

6. A vector field with zero curl is called

- (a) rotational (b) divergent (c) solenoidal (d) irrotational

☐ (d) — Can be written as $\nabla\phi$.

7. $\nabla \times (\nabla\phi)$ equals

- (a) 1 (b) $\nabla\phi$ (c) 0 (d) $\nabla^2\phi$

☐ (c) — Curl of gradient is zero.

8. $\nabla \cdot (\nabla \times \mathbf{A})$ equals

- (a) $\nabla \cdot \mathbf{A}$ (b) 0 (c) 1 (d) $\nabla^2\mathbf{A}$

☐ (b) — Divergence of curl is zero.



9. $\nabla \cdot \mathbf{r}$ for the position vector $\mathbf{r}$ is

- (a) 1 (b) 3 (c) r (d) 0

☐ (b) $-1 + 1 + 1$.

10. $\nabla \times \mathbf{r}$ equals

- (a) 0 (b) $\mathbf{r}$ (c) 3 (d) 1

☐ (a) — Position vector is irrotational.

11. The Laplacian of ϕ is

- (a) $\nabla\phi \times \nabla\phi$ (b) $\nabla\phi$ (c) $\nabla \times \nabla\phi$ (d) $\nabla \cdot \nabla\phi$

☐ (d) $-\nabla^2\phi$.

12. Gauss's divergence theorem relates

- (a) surface and volume integrals (b) line and surface integrals (c) two line integrals (d) volume and line integrals

☐ (a) — Flux $\leftrightarrow$ divergence.

13. Stokes' theorem relates

- (a) point and line (b) line and surface integrals (c) two volume integrals (d) surface and volume integrals

☐ (b) — Circulation $\leftrightarrow$ curl.

14. Maxwell's first equation is

- (a) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ (b) $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ (c) $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ (d) $\nabla \cdot \mathbf{B} = 0$

☐ (c) — Gauss's law.

15. $\nabla \cdot \mathbf{B} = 0$ means

- (a) $\mathbf{B}$ is zero (b) magnetic monopoles do not exist (c) $\mathbf{B}$ is uniform (d) $\mathbf{B}$ is irrotational

☐ (b) — No isolated magnetic charges.

16. $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ represents

- (a) Coulomb's law (b) Faraday's law (c) Gauss's law (d) Ampère's law

☐ (b) — Changing B produces E.

17. The displacement current was introduced by

- (a) Gauss (b) Ampère (c) Faraday (d) Maxwell

☐ (d) — To make Ampère's law consistent.

18. Displacement current density is

- (a) $\sigma\mathbf{E}$ (b) $\epsilon_0\partial\mathbf{E}/\partial t$ (c) $\mu_0\partial\mathbf{B}/\partial t$ (d) $\rho\mathbf{v}$

☐ (b) — Due to changing E.

19. Displacement current exists

- (a) wherever the electric field changes with time (b) only in conductors (c) only in magnets (d) only in DC circuits

☐ (a) — Including vacuum.

20. In a charging capacitor, displacement current between plates equals

- (a) twice the conduction current (b) half the conduction current (c) zero (d) conduction current in wires

☐ (d) — Continuity.

21. The speed of EM waves in vacuum is

- (a) $1/\sqrt{(\mu_0\epsilon_0)}$ (b) $\sqrt{(\mu_0\epsilon_0)}$ (c) $\mu_0\epsilon_0$ (d) $1/(\mu_0\epsilon_0)$

☐ (a) — From the wave equation.

22. EM waves are

- (a) transverse (b) longitudinal (c) both (d) neither

☐ (a) — $\mathbf{E}, \mathbf{B} \perp$ direction.



- 23.** In an EM wave, **E** and **B** are
(a) perpendicular and in phase (b) parallel and out of phase (c) parallel and in phase (d) perpendicular and 90° out of phase
□ (a) — Plane wave.
- 24.** The ratio E/B in an EM wave in vacuum is
(a) 1/c (b) 1 (c) c² (d) c
□ (d) — $E = cB$.
- 25.** The direction of propagation of an EM wave is along
(a) **E** (b) **B** (c) **B** × **E** (d) **E** × **B**
□ (d) — Poynting vector.
- 26.** The Poynting vector represents
(a) charge density (b) potential (c) momentum only (d) energy flux (power per unit area)
□ (d) — $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$.
- 27.** The refractive index of a medium in terms of μ_r, ϵ_r is
(a) $1/\sqrt{(\mu_r \epsilon_r)}$ (b) μ_r/ϵ_r (c) $\sqrt{(\mu_r \epsilon_r)}$ (d) $\mu_r \epsilon_r$
□ (c) — $n = c/v$.
- 28.** In a conducting medium, EM waves are
(a) attenuated (b) amplified (c) converted to sound (d) unchanged
□ (a) — Energy lost as Joule heat.
- 29.** Skin depth is inversely proportional to
(a) 1/σ (b) σ (c) σ² (d) √σ
□ (d) — $\delta = \sqrt{2/\mu\sigma\omega}$.
- 30.** As frequency increases, skin depth
(a) increases (b) decreases (c) stays constant (d) becomes infinite
□ (b) — $\delta \propto 1/\sqrt{\omega}$.
- 31.** Skin depth is the distance in which the amplitude falls to
(a) zero (b) 1/e (c) 1/e² (d) 1/2
□ (b) — Definition.
- 32.** The amplitude of an EM wave in a conductor varies as
(a) $e^{-z/\delta}$ (b) z (c) $e^{z/\delta}$ (d) 1/z
□ (a) — Exponential decay.
- 33.** For a good conductor,
(a) $\sigma \ll \omega\epsilon$ (b) $\epsilon = 0$ (c) $\sigma = 0$ (d) $\sigma \gg \omega\epsilon$
□ (d) — Conduction dominates.
- 34.** The energy density of an EM wave is shared
(a) equally by E and B (b) only by B (c) mostly by B (d) only by E
□ (a) — $(1/2)\epsilon_0 E^2 = B^2/2\mu_0$.
- 35.** Faraday's law in integral form is
(a) $\oint \mathbf{E} \cdot d\mathbf{S} = q/\epsilon_0$ (b) $\oint \mathbf{B} \cdot d\mathbf{S} = 0$ (c) $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$ (d) $\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi_B/dt$
□ (d) — Induced emf.
- 36.** The equation of continuity is
(a) $\mathbf{J} = \sigma \mathbf{E}$ (b) $\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0$ (c) $\nabla \cdot \mathbf{J} = \rho$ (d) $\nabla \times \mathbf{J} = 0$
□ (b) — Charge conservation.
- 37.** The electric field is related to the potential by
(a) $\mathbf{E} = \nabla^2 V$ (b) $\mathbf{E} = \nabla \times V$ (c) $\mathbf{E} = -\nabla V$ (d) $\mathbf{E} = \nabla V$
□ (c) — Field points to lower potential.



38. The flux of **B** through any closed surface is

- (a) zero (b) negative (c) infinite (d) positive

□ (a) — Gauss's law for magnetism.

39. Maxwell's equations predict that light is

- (a) a sound wave (b) a longitudinal wave (c) a particle stream (d) an electromagnetic wave

□ (d) — $v = c$.

40. The unit of ϵ_0 is

- (a) V/m (b) F/m (c) C/m (d) H/m

□ (b) — Farad per metre.

Module IV: Quantum Physics

Classical physics could not explain black-body radiation, the photoelectric effect, atomic spectra or the stability of atoms. Quantum physics replaced it at the atomic scale. Its key idea is that matter and radiation both have a dual wave-particle nature, and that physical quantities such as energy often take only discrete values. Semiconductors, lasers, LEDs, transistors and MRI all depend on quantum physics.

4.1 Wave-particle duality

Radiation behaves as particles.

- **Black-body radiation (Planck, 1900):** energy is emitted and absorbed in discrete packets (quanta) $E = h\nu$.
- **Photoelectric effect (Einstein, 1905):** light consists of photons of energy $h\nu$. Electrons are emitted only if $h\nu$ exceeds the work function ϕ : $KE_{\max} = h\nu - \phi$. The emission is instantaneous and $KE_{\max}$ depends on frequency, not intensity.
- **Compton effect (1923):** X-rays scattered by electrons have a longer wavelength, $\Delta\lambda = (h/m_0c)(1 - \cos \theta)$. This can only be explained by treating the photon as a particle with momentum $p = h/\lambda$.

Photon properties.

$$E = h\nu = hc/\lambda, p = h/\lambda = E/c, \text{ rest mass} = 0, \text{ speed} = c$$
$$E \text{ (in eV)} = 12400/\lambda \text{ (in \AA)} \text{ (useful shortcut)}$$

Matter behaves as waves. Electrons show diffraction (Davisson–Germer experiment, 1927; G.P. Thomson). So both radiation and matter show wave and particle properties. Which aspect appears depends on the experiment; the two aspects are never observed simultaneously in a single measurement (Bohr's complementarity).

4.2 de Broglie hypothesis: matter waves

In 1924 de Broglie proposed that every moving particle has a wave associated with it, with wavelength

$$\lambda = h/p = h/(mv)$$

Different forms of the de Broglie wavelength.

In terms of kinetic energy E : $\lambda = h/\sqrt{2mE}$
Charged particle accelerated through V volts: $\lambda = h/\sqrt{2meV}$
For an electron: $\lambda = 12.27/\sqrt{V} \text{ \AA}$ (V in volts)
Thermal particle at temperature T : $\lambda = h/\sqrt{3mk_B T}$

Why we don't notice matter waves for everyday objects. A 0.1 kg ball moving at 10 m/s has $\lambda = 6.63 \times 10^{-34}/1 = 6.6 \times 10^{-34} \text{ m}$, far too small to detect. An electron accelerated through 100 V has $\lambda \approx 1.23 \text{ \AA}$, comparable to atomic spacings, so electron diffraction by crystals is observed.

Properties of matter waves.

- They are not electromagnetic; they are associated with any moving particle, charged or not.

- λ is smaller for heavier or faster particles.
- Their phase velocity can exceed c (see next section); this does not violate relativity because they carry no energy at that speed.
- They are described by a wave function ψ whose meaning is statistical (Section 4.5).

Applications: electron microscope (resolution far better than optical, since λ is very small), neutron diffraction for crystal structure.

4.3 Phase velocity and group velocity

A single harmonic wave $y = A \sin(\omega t - kx)$ moves with the **phase velocity** (wave velocity)

$$v_p = \omega/k$$

A single infinite wave cannot represent a localised particle. A particle is represented by a **wave packet** (wave group): a superposition of many waves with slightly different wavelengths, which reinforce in a small region and cancel elsewhere. The envelope of the packet moves with the **group velocity**

$$v_g = d\omega/dk$$

Superposition of two waves. $y_1 = A \sin(\omega t - kx)$, $y_2 = A \sin[(\omega + \Delta\omega)t - (k + \Delta k)x]$. Adding: $y = 2A \cos[(\Delta\omega/2)t - (\Delta k/2)x] \sin(\omega t - kx)$ (approximately). The fast carrier moves at ω/k ; the slowly varying envelope (the group, the "beat") moves at $\Delta\omega/\Delta k \rightarrow d\omega/dk$.

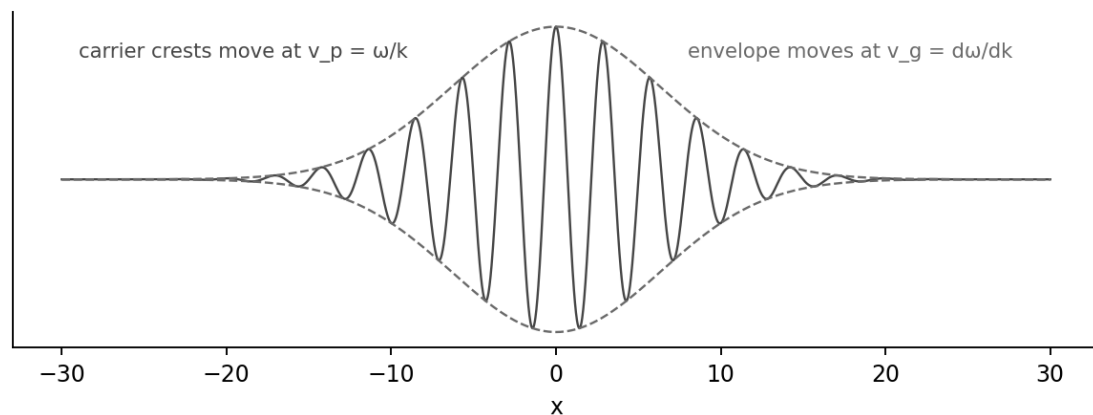


Figure 4.1: A wave packet. The crests move at the phase velocity; the envelope (the particle) moves at the group velocity.

Relation between phase and group velocity. Since $\omega = v_p k$,

$$v_g = v_p + k \, dv_p/dk = v_p - \lambda \, dv_p/d\lambda$$

- In a **non-dispersive** medium (v_p independent of λ , e.g. light in vacuum): $v_g = v_p$.
- In a **normally dispersive** medium ($dv_p/d\lambda > 0$): $v_g < v_p$.

For a de Broglie wave of a free non-relativistic particle. $E = \hbar\omega$ and $p = \hbar k$, with $E = p^2/2m$. So $\omega = \hbar k^2/2m$.

$$v_g = d\omega/dk = \hbar k/m = p/m = v \text{ (particle velocity); } v_p = \omega/k = \hbar k/2m = v/2$$

Relativistically, $v_p v_g = c^2$, so $v_p = c^2/v > c$. In every case the **group velocity equals the particle velocity**. This is why a particle is identified with the wave packet, not the individual wave.

4.4 Heisenberg's uncertainty principle

Because a particle is a wave packet of finite size, its position and momentum cannot both be known exactly at the same time:

$$\Delta x \Delta p_x \geq \hbar/2 \text{ (often written } \Delta x \Delta p \approx \hbar)$$

$$\Delta E \Delta t \geq \hbar/2, \Delta L \Delta \theta \geq \hbar/2$$

A narrow packet (small Δx) needs many wavelengths (large Δk , hence large Δp); a packet with nearly one wavelength (small Δp) must be very long (large Δx).

Consequences. Electrons cannot exist inside the nucleus (confinement to 10^{-14} m would need kinetic energies of at least about 10 MeV, while beta-decay electrons have energies of only a few MeV); an oscillator has a non-zero minimum (zero-point) energy; spectral lines have a natural width.

4.5 Wave function and its physical meaning

The state of a quantum particle is described by a **wave function** $\psi(x, t)$, generally a complex function. ψ itself has no direct physical meaning and cannot be measured.

Born's interpretation (1926). $|\psi(x, t)|^2 = \psi^* \psi$ is the **probability density**: $|\psi|^2 dx$ is the probability of finding the particle between x and $x + dx$ at time t . In three dimensions, $|\psi|^2 dV$ is the probability of finding it in the volume dV .

Conditions for an acceptable (well-behaved) wave function.

- ψ must be **single-valued** (only one probability at each point).
- ψ must be **finite** everywhere (probability cannot be infinite).
- ψ and its first derivative $d\psi/dx$ must be **continuous** (except where the potential is infinite).
- ψ must be **square-integrable** (normalisable): $\int |\psi|^2 dx$ is finite.

4.6 Normalisation

Since the particle must be somewhere, the total probability over all space is 1:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \text{ (normalisation condition)}$$

If a solution ψ is not normalised, multiply it by a constant N chosen so that $N^2 \int |\psi|^2 dx = 1$. N is the **normalisation constant**.

Example. $\psi = A e^{-|x|/a}$: $\int |\psi|^2 dx = A^2 \times 2 \int_0^{\infty} e^{-2x/a} dx = A^2 a = 1 \Rightarrow A = 1/\sqrt{a}$.

Orthogonality. Two wave functions ψ_m and ψ_n ($m \neq n$) of the same system are orthogonal if $\int \psi_m^* \psi_n dx = 0$. Normalised and mutually orthogonal wave functions form an **orthonormal set**: $\int \psi_m^* \psi_n dx = \delta_{mn}$.

4.7 Observables as operators

In quantum mechanics every measurable physical quantity (an **observable**) is represented by a linear operator acting on ψ .

Observable	Classical form	Quantum operator
Position	x	$\hat{x} = x$
Momentum	p_x	$\hat{p}_x = -i\hbar \partial/\partial x$
Kinetic energy	$p^2/2m$	$-(\hbar^2/2m) \partial^2/\partial x^2$
Potential energy	$V(x)$	$V(x)$
Total energy (Hamiltonian)	$H = p^2/2m + V$	$\hat{H} = -(\hbar^2/2m)\nabla^2 + V$
Total energy (time)	E	$\hat{E} = i\hbar \partial/\partial t$

Eigenvalue equation. If an operator $\hat{A}$ acting on a function ψ gives a constant multiple of the same function,

$$\hat{A}\psi = a\psi$$

then ψ is an **eigenfunction** of $\hat{A}$ and a is the corresponding **eigenvalue**. When the system is in an eigenstate of $\hat{A}$, a measurement of the observable always gives exactly the value a . (Compare with matrix eigenvalues $AX = \lambda X$ in Mathematics-I, Module V.)

Example. $\psi = e^{ikx}$: $\hat{p}\psi = -i\hbar(ik)e^{ikx} = \hbar k \psi$. So e^{ikx} is an eigenfunction of momentum with eigenvalue $p = \hbar k = h/\lambda$ (de Broglie!). But $\sin kx$ is not an eigenfunction of $\hat{p}$ (the derivative gives $\cos kx$); it is an eigenfunction of $\hat{p}^2$ with eigenvalue $\hbar^2 k^2$.

Hermitian operators. Operators that represent observables are Hermitian, so their eigenvalues are real (like real symmetric matrices).

4.8 Expectation values

When a system is not in an eigenstate of $\hat{A}$, individual measurements give different results. The average of many measurements on identically prepared systems is the **expectation value**:

$$\langle A \rangle = \int \psi^* \hat{A} \psi \, dx \text{ (with } \psi \text{ normalised)}$$

$$\langle x \rangle = \int x |\psi|^2 \, dx, \langle p \rangle = \int \psi^* (-i\hbar \, d\psi/dx) \, dx, \langle E \rangle = \int \psi^* \hat{H} \psi \, dx$$

The uncertainty in A is $\Delta A = \sqrt{(\langle A^2 \rangle - \langle A \rangle^2)}$.

4.9 Schrödinger's time-dependent equation

Start from a free-particle wave $\psi = A e^{i(kx - \omega t)} = A e^{(i/\hbar)(px - Et)}$. Then $\partial\psi/\partial t = -(iE/\hbar)\psi \Rightarrow E\psi = i\hbar \partial\psi/\partial t$, and $\partial^2\psi/\partial x^2 = -(p^2/\hbar^2)\psi \Rightarrow p^2\psi = -\hbar^2 \partial^2\psi/\partial x^2$. The total energy is $E = p^2/2m + V$. Multiply by ψ and substitute:

$$i\hbar \partial\psi/\partial t = -(\hbar^2/2m) \partial^2\psi/\partial x^2 + V(x, t) \psi \text{ (i.e. } \hat{E}\psi = \hat{H}\psi)$$

In three dimensions, $i\hbar \partial\psi/\partial t = -(\hbar^2/2m)\nabla^2\psi + V\psi$. This is the fundamental equation of non-relativistic quantum mechanics; it plays the role that Newton's second law plays in classical mechanics.

4.10 Schrödinger's time-independent equation

If V depends only on position, separate variables: $\psi(x, t) = \varphi(x) e^{-iEt/\hbar}$. Substituting in the time-dependent equation, the time factor cancels and

$$-(\hbar^2/2m) d^2\varphi/dx^2 + V(x) \varphi = E \varphi \text{ or } d^2\varphi/dx^2 + (2m/\hbar^2)(E - V) \varphi = 0$$

This is an eigenvalue equation $\hat{H}\psi = E\psi$: the allowed energies E are the eigenvalues of the Hamiltonian. States of this form are **stationary states**: $|\psi|^2 = |\phi|^2$ does not change with time. For a free particle ($V = 0$): $d^2\psi/dx^2 + k^2\psi = 0$ with $k^2 = 2mE/\hbar^2$.

4.11 Particle in a one-dimensional box (infinite potential well)

A particle of mass m moves freely along x between two rigid walls at $x = 0$ and $x = L$:

$$V(x) = 0 \text{ for } 0 < x < L; V(x) = \infty \text{ for } x \leq 0 \text{ and } x \geq L$$

Outside the box the particle cannot exist (infinite potential), so $\psi = 0$ there. **Inside**, Schrödinger's equation is $d^2\psi/dx^2 + k^2\psi = 0$ with $k^2 = 2mE/\hbar^2$. The general solution is $\psi = A \sin kx + B \cos kx$.

Boundary conditions. Continuity of ψ requires $\psi(0) = 0$ and $\psi(L) = 0$.

- $\psi(0) = 0 \Rightarrow B = 0$, so $\psi = A \sin kx$.
- $\psi(L) = 0 \Rightarrow A \sin kL = 0$. $A \neq 0$ (else no particle), so $kL = n\pi$, i.e. $k = n\pi/L$, $n = 1, 2, 3, \dots$ ($n = 0$ gives $\psi = 0$ everywhere, which is not allowed).

Energy eigenvalues. From $k^2 = 2mE/\hbar^2$:

$$E_n = n^2\pi^2\hbar^2/(2mL^2) = n^2h^2/(8mL^2), n = 1, 2, 3, \dots$$

Normalised eigenfunctions. $\int_0^L A^2 \sin^2(n\pi x/L) dx = A^2L/2 = 1 \Rightarrow A = \sqrt{2/L}$.

$$\psi_n(x) = \sqrt{2/L} \sin(n\pi x/L), 0 < x < L$$

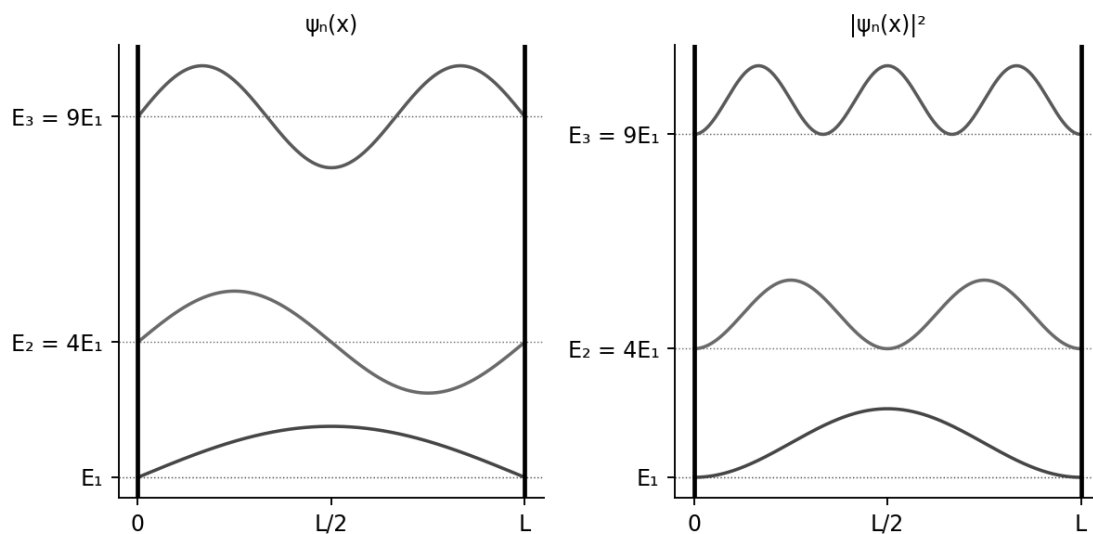


Figure 4.2: Particle in a box: wave functions ψ_n and probability densities $|\psi_n|^2$ for $n = 1, 2, 3$, drawn at their energy levels.

Important results.

- **Energy is quantised.** Only discrete energies are allowed; $E_n \propto n^2$. $E_1 : E_2 : E_3 = 1 : 4 : 9$.
- **Zero-point energy.** The lowest energy $E_1 = h^2/(8mL^2)$ is not zero. The particle can never be at rest, in agreement with the uncertainty principle.
- **Energy spacing** $E_{n+1} - E_n = (2n + 1)h^2/(8mL^2)$ grows with n and decreases as L or m increases. For macroscopic L and m the levels are so close that energy appears

continuous (the classical limit).

- **Nodes.** ψ_n has $(n - 1)$ nodes inside the box. The probability density $|\psi_n|^2 = (2/L)\sin^2(n\pi x/L)$ is not uniform: for $n = 1$ the particle is most likely at the centre; for $n = 2$ the probability at the centre is zero. Classically, the probability would be uniform.
- **Standing waves.** The allowed states are standing de Broglie waves with $L = n\lambda/2$, just like the modes of a string fixed at both ends.
- **Momentum.** $p = \pm\hbar k = \pm nh/(2L)$; the particle moves back and forth.

Applications. Free electrons in a metal (free electron model), conjugated dye molecules, quantum wells and quantum dots in semiconductor lasers and LEDs (smaller dots emit bluer light because the energy gap $\propto 1/L^2$).

Three-dimensional box of sides a, b, c : $E = (h^2/8m)(n_x^2/a^2 + n_y^2/b^2 + n_z^2/c^2)$. For a cube, different sets (n_x, n_y, n_z) can give the same energy: **degeneracy**. For example $(1,1,2)$, $(1,2,1)$, $(2,1,1)$ all give $E = 6h^2/(8ma^2)$: a threefold degenerate level.

4.12 Common mistakes to avoid in Module IV

- Using $\lambda = h/mv$ with the mass in grams or the energy in eV without converting to SI units.
- Taking $n = 0$ in the particle-in-a-box problem. The lowest state is $n = 1$.
- Writing $E_n \propto n$ instead of n^2 .
- Saying ψ is the probability. The probability density is $|\psi|^2$.
- Confusing phase velocity (ω/k) and group velocity ($d\omega/dk$). The particle moves with the group velocity.

4.13 Failure of classical physics (background for long answers)

Phenomenon	Classical prediction	Observation	Quantum explanation
Black-body radiation	Rayleigh-Jeans law: energy $\rightarrow \infty$ at short wavelengths ("ultraviolet catastrophe")	Spectrum peaks and falls to zero at short λ	Planck: energy quantised in units $h\nu$
Photoelectric effect	KE of electrons should depend on intensity; time delay expected	KE depends on frequency; threshold frequency; no time delay	Einstein: light consists of photons of energy $h\nu$
Compton scattering	Scattered X-rays should have the same wavelength	Wavelength increases with scattering angle	Photon-electron collision with $p = h/\lambda$
Atomic spectra	Continuous spectrum; atoms should collapse	Sharp lines; stable atoms	Quantised energy levels (Bohr, Schrödinger)
Specific heat of solids	Constant $3R$ at all temperatures	Falls to zero at low T	Quantised lattice vibrations (Einstein, Debye)

4.14 Derivation of de Broglie's relation

For a photon, $E = h\nu$ (Planck) and $E = pc$ (relativity, zero rest mass). So $p = h\nu/c = h/\lambda$, i.e. $\lambda = h/p$. de Broglie argued, by the symmetry of nature, that the same relation should

hold for material particles: $\lambda = h/(mv)$.

Bohr's condition from de Broglie waves. An electron in a stationary orbit of radius r forms a standing wave only if the circumference contains a whole number of wavelengths: $2\pi r = n\lambda = nh/(mv)$. Hence $mvr = nh/2\pi = n\hbar$, which is exactly Bohr's quantisation condition. This gave the first physical explanation of why only certain orbits are allowed.

Electron wavelength with relativistic correction. For high voltages, $\lambda = h/\sqrt{(2m_0 eV(1 + eV/(2m_0 c^2)))}$. At 100 kV (electron microscopes) the correction is about 5%.

4.15 Davisson-Germer experiment (verification of matter waves)

A beam of electrons accelerated through a potential V is directed onto a nickel crystal, and the intensity of scattered electrons is measured as a function of angle. A sharp maximum was found at 50° for $V = 54$ V.

- de Broglie wavelength: $\lambda = 12.27/\sqrt{54} = 1.67$ Å.
- Bragg reflection from atomic planes (spacing $d = 0.91$ Å): $2d \sin \theta = n\lambda$ with $\theta = 65^\circ$ (glancing angle): $\lambda = 2 \times 0.91 \times \sin 65^\circ = 1.65$ Å.

The agreement confirmed that electrons diffract like waves with $\lambda = h/p$.

4.16 Derivation of the uncertainty principle from a wave packet (qualitative)

A wave packet of length Δx can be built from waves whose wave numbers spread over Δk , with $\Delta x \Delta k \approx 2\pi$ (the more localised the packet, the more wave numbers are needed). Using $p = \hbar k$, $\Delta p = \hbar \Delta k$, so $\Delta x \Delta p \approx 2\pi\hbar = h$. The rigorous minimum (for a Gaussian packet) is $\Delta x \Delta p = \hbar/2$.

Thought experiment (Heisenberg's γ -ray microscope). To locate an electron with light of wavelength λ , the uncertainty in position is $\Delta x \approx \lambda/(2 \sin \alpha)$, where α is the half-angle of the microscope objective. The scattered photon, of momentum h/λ , may enter anywhere within the cone, so the electron's recoil momentum is uncertain by $\Delta p \approx 2(h/\lambda) \sin \alpha$. Product: $\Delta x \Delta p \approx h$. Improving position accuracy (shorter λ) increases the momentum kick.

Application: ground-state energy of a harmonic oscillator. $E = p^2/2m + (1/2)m\omega^2 x^2$. With $p \approx \hbar/(2x)$, $E \approx \hbar^2/(8mx^2) + (1/2)m\omega^2 x^2$. Minimising with respect to x gives $E_{\min} \approx \hbar\omega/2$, the exact zero-point energy.

Application: size of the hydrogen atom. $E \approx \hbar^2/(2mr^2) - e^2/(4\pi\epsilon_0 r)$. Minimising gives $r \approx 4\pi\epsilon_0 \hbar^2/(me^2) = 0.53$ Å (the Bohr radius) and $E \approx -13.6$ eV.

4.17 Properties of the Schrödinger equation and stationary states

- It is **linear and homogeneous** in ψ : if ψ_1 and ψ_2 are solutions, so is $c_1\psi_1 + c_2\psi_2$ (superposition principle).
- It is **first order in time**: knowing ψ at $t = 0$ determines ψ at all later times.
- It contains i , so ψ is in general complex; only $|\psi|^2$ has a direct physical meaning.
- **Stationary states** $\psi = \phi(x)e^{-iEt/\hbar}$ have a definite energy E and a time-independent probability density. Expectation values of all time-independent observables are constant in these states.
- The time-independent equation, together with boundary conditions, allows solutions only for certain energies: **quantisation arises naturally** from the requirement that ψ be well-behaved.

Probability current and conservation. From the Schrödinger equation, $\partial|\psi|^2/\partial t + \partial j/\partial x = 0$ with $j = (\hbar/2mi)(\psi^*\partial\psi/\partial x - \psi\partial\psi^*/\partial x)$. The total probability $\int |\psi|^2 dx$ stays equal to 1 for all times (just like charge conservation in Module III).

4.18 Particle in a box: summary table

Quantity	Formula
Allowed wave number	$k_n = n\pi/L$
Allowed wavelength	$\lambda_n = 2L/n$
Energy	$E_n = n^2\hbar^2/(8mL^2)$
Wave function	$\psi_n = \sqrt{2/L} \sin(n\pi x/L)$
Momentum magnitude	$p_n = nh/(2L)$
Energy gap between n and $n + 1$	$(2n + 1)\hbar^2/(8mL^2)$
Number of nodes inside	$n - 1$
$\langle x \rangle$	$L/2$
$\langle p \rangle$	0 (equally likely to move either way)
$\langle p^2 \rangle$	$n^2\hbar^2/(4L^2) = 2mE_n$
Ratio $E_1 : E_2 : E_3$	1 : 4 : 9

4.19 Solved numericals: Module IV

Problem 1. Find the energy (in J and eV) and momentum of a photon of wavelength 5000 Å.

$$E = hc/\lambda = 6.626 \times 10^{-34} \times 3 \times 10^8 / (5 \times 10^{-7}) = 3.98 \times 10^{-19} \text{ J} = 3.98 \times 10^{-19} / 1.602 \times 10^{-19} \text{ eV} = 2.48 \text{ eV.}$$

$$p = h/\lambda = 6.626 \times 10^{-34} / 5 \times 10^{-7} = 1.33 \times 10^{-27} \text{ kg m/s.}$$

Answer: $E \approx 2.48 \text{ eV}$ ($3.98 \times 10^{-19} \text{ J}$); $p \approx 1.33 \times 10^{-27} \text{ kg m/s}$.

Problem 2. Find the de Broglie wavelength of an electron accelerated through a potential difference of 100 V.

$$\lambda = h/\sqrt{2meV} = 6.626 \times 10^{-34} / \sqrt{2 \times 9.11 \times 10^{-31} \times 1.602 \times 10^{-19} \times 100}.$$

$$\text{Denominator} = \sqrt{2.919 \times 10^{-47}} = 5.403 \times 10^{-24}. \lambda = 1.226 \times 10^{-10} \text{ m.}$$

$$\text{Check with the shortcut: } 12.27/\sqrt{100} = 1.227 \text{ Å.}$$

Answer: $\lambda \approx 1.23 \text{ Å}$.

Problem 3. Calculate the de Broglie wavelength of a cricket ball of mass 0.1 kg moving at 10 m/s. Comment.

$$\lambda = h/mv = 6.626 \times 10^{-34} / (0.1 \times 10).$$

Answer: $\lambda = 6.6 \times 10^{-34} \text{ m}$. This is far too small to detect, so wave properties of everyday objects are never observed.

Problem 4. Find the de Broglie wavelength of an electron with kinetic energy 1 keV.

$$\lambda = h/\sqrt{2mE} = 6.626 \times 10^{-34} / \sqrt{2 \times 9.11 \times 10^{-31} \times 1000 \times 1.602 \times 10^{-19}}.$$



$$\sqrt{(2.919 \times 10^{-46})} = 1.708 \times 10^{-23}.$$

Answer: $\lambda \approx 3.88 \times 10^{-11} \text{ m} = 0.388 \text{ \AA}$.

Problem 5. Find the de Broglie wavelength of a thermal neutron at 300 K ($m = 1.675 \times 10^{-27} \text{ kg}$).

$$\lambda = h/\sqrt{3mk_B T} = 6.626 \times 10^{-34}/\sqrt{3 \times 1.675 \times 10^{-27} \times 1.38 \times 10^{-23} \times 300}.$$

$$\sqrt{(2.080 \times 10^{-47})} = 4.561 \times 10^{-24}.$$

Answer: $\lambda \approx 1.45 \text{ \AA}$ (comparable to interatomic spacing, so thermal neutrons are diffracted by crystals).

Problem 6. A proton and an electron have the same de Broglie wavelength. Which has more kinetic energy, and by what factor?

Same $\lambda \Rightarrow$ same momentum p . $KE = p^2/2m$, so $KE \propto 1/m$.

$$KE_e/KE_p = m_p/m_e = 1.673 \times 10^{-27}/9.11 \times 10^{-31} = 1836.$$

Answer: The electron has about 1836 times more kinetic energy.

Problem 7. Through what potential difference must an electron be accelerated to have a de Broglie wavelength of $0.5 \text{ \AA}$?

$$\lambda = 12.27/\sqrt{V} \text{ \AA} \Rightarrow \sqrt{V} = 12.27/0.5 = 24.54 \Rightarrow V = 602 \text{ V}.$$

Answer: About 602 V.

Problem 8. Show that for a free non-relativistic particle the group velocity equals the particle velocity and the phase velocity is half of it.

$$E = \hbar\omega = p^2/2m \text{ and } p = \hbar k \Rightarrow \omega = \hbar k^2/(2m).$$

$$v_g = d\omega/dk = \hbar k/m = p/m = v. \quad v_p = \omega/k = \hbar k/(2m) = v/2.$$

Answer: $v_g = v$; $v_p = v/2$.

Problem 9. The dispersion relation of water waves in deep water is $\omega = \sqrt{gk}$. Find the ratio of group velocity to phase velocity.

$$v_p = \omega/k = \sqrt{g/k}. \quad v_g = d\omega/dk = (1/2)\sqrt{g/k}.$$

Answer: $v_g/v_p = 1/2$.

Problem 10. An electron is confined to a region of length $1 \text{ \AA}$. Find the minimum uncertainty in its momentum and velocity.

$$\Delta p \geq \hbar/(2\Delta x) = 1.055 \times 10^{-34}/(2 \times 10^{-10}) = 5.28 \times 10^{-25} \text{ kg m/s}.$$

$$\Delta v = \Delta p/m = 5.28 \times 10^{-25}/9.11 \times 10^{-31} = 5.8 \times 10^5 \text{ m/s}.$$

Answer: $\Delta p \approx 5.3 \times 10^{-25} \text{ kg m/s}$; $\Delta v \approx 5.8 \times 10^5 \text{ m/s}$.

Problem 11. The lifetime of an excited atomic state is 10^{-8} s . Find the minimum uncertainty in its energy and the natural width of the emitted line in frequency.

$$\Delta E \geq \hbar/(2\Delta t) = 1.055 \times 10^{-34}/(2 \times 10^{-8}) = 5.28 \times 10^{-27} \text{ J} \approx 3.3 \times 10^{-8} \text{ eV}.$$

$$\Delta\nu = \Delta E/h = 5.28 \times 10^{-27}/6.626 \times 10^{-34} \approx 8 \times 10^6 \text{ Hz.}$$

Answer: $\Delta E \approx 3.3 \times 10^{-8} \text{ eV}$; $\Delta\nu \approx 8 \text{ MHz}$.

Problem 12. An electron is in a one-dimensional box of length 1 Å. Find the first three energy levels in eV.

$$E_1 = h^2/(8mL^2) = (6.626 \times 10^{-34})^2/(8 \times 9.11 \times 10^{-31} \times 10^{-20}) = 4.39 \times 10^{-67}/7.288 \times 10^{-50} = 6.02 \times 10^{-18} \text{ J} = 37.6 \text{ eV.}$$

$$E_2 = 4E_1 = 150.4 \text{ eV}; E_3 = 9E_1 = 338.4 \text{ eV.}$$

Answer: 37.6 eV, 150.4 eV, 338.4 eV.

Problem 13. An electron is in a box of width 1 nm. Find the ground-state energy and the wavelength of the photon emitted in the transition from $n = 2$ to $n = 1$.

$$E_1 = 37.6 \text{ eV} \times (1 \text{ Å}/1 \text{ nm})^2 = 37.6/100 = 0.376 \text{ eV.}$$

$$\Delta E = E_2 - E_1 = 3E_1 = 1.128 \text{ eV. } \lambda = 12400/1.128 \text{ Å} \approx 11000 \text{ Å.}$$

Answer: $E_1 \approx 0.376 \text{ eV}$; $\lambda \approx 1.1 \text{ } \mu\text{m}$ (near infrared).

Problem 14. A 10 g marble is confined in a box of length 10 cm. Find its zero-point energy. Why is quantisation not observed?

$$E_1 = h^2/(8mL^2) = (6.626 \times 10^{-34})^2/(8 \times 0.01 \times 0.01) = 4.39 \times 10^{-67}/8 \times 10^{-4}.$$

Answer: $E_1 \approx 5.5 \times 10^{-64} \text{ J}$. The levels are so close together that the energy appears continuous.

Problem 15. For a particle in a box of length L in its ground state, find the probability of finding it between $x = 0$ and $x = L/4$.

$$P = (2/L) \int_0^{L/4} \sin^2(\pi x/L) dx = (2/L) [x/2 - (L/4\pi) \sin(2\pi x/L)]_0^{L/4} = (2/L) [L/8 - L/(4\pi)].$$

Answer: $P = 1/4 - 1/(2\pi) \approx 0.091$ (about 9%).

Problem 16. Show that the wave functions of a particle in a box are normalised and find $\langle x \rangle$ for any state.

$$\int_0^L (2/L) \sin^2(n\pi x/L) dx = (2/L)(L/2) = 1. \text{ Normalised.}$$

$$\langle x \rangle = (2/L) \int_0^L x \sin^2(n\pi x/L) dx = (2/L)(L^2/4) = L/2 \text{ (by symmetry about the centre).}$$

Answer: $\langle x \rangle = L/2$ for every state.

Problem 17. Normalise $\psi(x) = A e^{-a|x|}$ and find the probability of finding the particle between $-1/a$ and $1/a$.

$$\int_{-\infty}^{\infty} A^2 e^{-2a|x|} dx = 2A^2/(2a) = A^2/a = 1 \Rightarrow A = \sqrt{a}.$$

$$P = 2a \int_0^{1/a} e^{-2ax} dx = 1 - e^{-2}.$$

Answer: $A = \sqrt{a}$; $P \approx 0.865$.

Problem 18. Show that $\psi = \sin kx$ is not an eigenfunction of the momentum operator but is an eigenfunction of $\hat{p}^2$.

$$\hat{p} \psi = -i\hbar d(\sin kx)/dx = -i\hbar k \cos kx, \text{ not a multiple of } \sin kx.$$

$$\hat{p}^2 \psi = -\hbar^2 d^2(\sin kx)/dx^2 = \hbar^2 k^2 \sin kx.$$

Answer: Eigenfunction of $\hat{p}^2$ with eigenvalue $\hbar^2 k^2$ (kinetic energy $\hbar^2 k^2/2m$), but not of $\hat{p}$.

Problem 19. Find the degeneracy of the level $E = 14\hbar^2/(8ma^2)$ for a particle in a cubical box.

$$n_x^2 + n_y^2 + n_z^2 = 14 \Rightarrow \{1, 2, 3\} \text{ in any order: } 3! = 6 \text{ permutations.}$$

Answer: Six-fold degenerate.

4.20 More solved numericals: Module IV

Problem 1. The work function of sodium is 2.3 eV. Find the threshold wavelength and the maximum KE of photoelectrons emitted by light of 4000 Å.

$$\lambda_0 = 12400/2.3 \text{ Å} = 5391 \text{ Å}.$$

$$\text{Photon energy at } 4000 \text{ Å} = 12400/4000 = 3.1 \text{ eV. } KE_{\max} = 3.1 - 2.3 = 0.8 \text{ eV}.$$

Answer: $\lambda_0 \approx 5391 \text{ Å}$; $KE_{\max} = 0.8 \text{ eV}$.

Problem 2. X-rays of wavelength 0.5 Å are Compton scattered at 90°. Find the wavelength of the scattered X-rays.

$$\Delta\lambda = (h/m_0c)(1 - \cos 90^\circ) = 0.0243 \text{ Å (Compton wavelength of the electron).}$$

Answer: $\lambda' = 0.5243 \text{ Å}$.

Problem 3. Calculate the de Broglie wavelength of an α -particle ($m = 6.64 \times 10^{-27} \text{ kg}$) accelerated through 1000 V.

$$\lambda = h/\sqrt{(2mqV)}, q = 2e: \sqrt{(2 \times 6.64 \times 10^{-27} \times 3.204 \times 10^{-19} \times 1000)} = \sqrt{(4.255 \times 10^{-42})} = 2.063 \times 10^{-21}.$$

Answer: $\lambda \approx 3.21 \times 10^{-13} \text{ m}$.

Problem 4. Find the energy of an electron and of a photon, each of wavelength 1 Å.

$$\text{Electron: } E = h^2/(2m\lambda^2) = (6.626 \times 10^{-34})^2/(2 \times 9.11 \times 10^{-31} \times 10^{-20}) = 2.41 \times 10^{-17} \text{ J} = 150 \text{ eV}.$$

$$\text{Photon: } E = 12400 \text{ eV}.$$

Answer: Electron $\approx 150 \text{ eV}$; photon $\approx 12.4 \text{ keV}$ (for the same wavelength, the photon has much more energy).

Problem 5. The velocity of an electron (mass $9.11 \times 10^{-31} \text{ kg}$) is measured as 300 m/s with an uncertainty of 0.01%. Find the minimum uncertainty in its position.

$$\Delta v = 0.0001 \times 300 = 0.03 \text{ m/s. } \Delta p = m\Delta v = 2.733 \times 10^{-32} \text{ kg m/s.}$$

$$\Delta x \geq \hbar/(2\Delta p) = 1.055 \times 10^{-34}/(5.466 \times 10^{-32}).$$



Answer: $\Delta x \geq 1.93 \times 10^{-3} \text{ m} \approx 1.9 \text{ mm}$.

Problem 6. Using the uncertainty principle, estimate the minimum kinetic energy of an electron confined within an atom of size 10^{-10} m .

$$\Delta p \approx \hbar/(2\Delta x) = 5.28 \times 10^{-25} \text{ kg m/s. } KE \approx (\Delta p)^2/2m = 2.78 \times 10^{-49}/(1.822 \times 10^{-30}) \\ = 1.53 \times 10^{-19} \text{ J.}$$

Answer: $KE \approx 0.95 \text{ eV}$ (order of a few eV, consistent with atomic energies).

Problem 7. Find the phase and group velocities of an electron of velocity $2 \times 10^6 \text{ m/s}$ (non-relativistic).

$$v_g = v = 2 \times 10^6 \text{ m/s. Non-relativistically } v_p = v/2 = 10^6 \text{ m/s (using } E = p^2/2m).$$

$$\text{If the rest energy is included (} E = mc^2), v_p = c^2/v = 9 \times 10^{16}/2 \times 10^6 = 4.5 \times 10^{10} \text{ m/s.}$$

Answer: $v_g = 2 \times 10^6 \text{ m/s}$; $v_p = 10^6 \text{ m/s}$ (kinetic energy only) or $4.5 \times 10^{10} \text{ m/s}$ (with rest energy). Either way the particle moves at v_g .

Problem 8. The dispersion relation of a medium is $\omega = ak + bk^3$. Find the phase and group velocities.

$$v_p = \omega/k = a + bk^2. v_g = d\omega/dk = a + 3bk^2.$$

Answer: $v_p = a + bk^2$; $v_g = a + 3bk^2$.

Problem 9. An electron is confined in a 1-D box of length $2 \text{ \AA}$. Find the energy required to excite it from the ground state to the second excited state.

$$E_1 = 37.6/4 = 9.40 \text{ eV (since } E \propto 1/L^2). \text{ Second excited state } n = 3: E_3 = 9E_1.$$

$$\Delta E = 8E_1 = 75.2 \text{ eV.}$$

Answer: 75.2 eV .

Problem 10. Find the probability that a particle in a box of length L in the first excited state ($n = 2$) is found between $x = 0$ and $x = L/4$.

$$P = (2/L) \int_0^{L/4} \sin^2(2\pi x/L) dx = (2/L) [x/2 - (L/8\pi) \sin(4\pi x/L)]_0^{L/4} = (2/L)(L/8 - 0).$$

Answer: $P = 1/4$.

Problem 11. Find the expectation value of momentum $\langle p \rangle$ for $\psi = \sqrt{(2/L)} \sin(\pi x/L)$.

$$\langle p \rangle = \int \psi^* (-i\hbar d\psi/dx) dx = -i\hbar(2/L)(\pi/L) \int_0^L \sin(\pi x/L) \cos(\pi x/L) dx = 0 \text{ (the integral of } \sin 2\theta \text{ over a full period of } \theta \text{ is zero).}$$

Answer: $\langle p \rangle = 0$.

Problem 12. Find $\langle x^2 \rangle$ for the ground state of a particle in a box and hence Δx .

$$\langle x^2 \rangle = (2/L) \int_0^L x^2 \sin^2(\pi x/L) dx = L^2(1/3 - 1/(2\pi^2)) = 0.2827L^2.$$

$$\Delta x = \sqrt{(\langle x^2 \rangle - \langle x \rangle^2)} = \sqrt{(0.2827 - 0.25)} L = 0.181L.$$

Answer: $\langle x^2 \rangle \approx 0.283L^2$; $\Delta x \approx 0.18L$.



Problem 13. Verify the uncertainty principle for the ground state of a particle in a box.

$$\Delta x = 0.181L \text{ (previous problem). } \langle p \rangle = 0, \langle p^2 \rangle = h^2/(4L^2) \Rightarrow \Delta p = h/(2L) = \pi\hbar/L.$$

$$\Delta x \Delta p = 0.181 \times \pi \hbar = 0.568\hbar.$$

Answer: $0.568\hbar > 0.5\hbar$, consistent with Heisenberg's principle.

4.21 Practice problems with solutions: Module IV

Q1. Find the energy of an X-ray photon of wavelength 1 Å in keV.

$$E = 12400/1 \text{ eV} = 12400 \text{ eV}.$$

Answer: 12.4 keV.

Q2. Find the de Broglie wavelength of an electron moving at 10^6 m/s.

$$\lambda = h/mv = 6.626 \times 10^{-34} / (9.11 \times 10^{-31} \times 10^6) = 7.27 \times 10^{-10} \text{ m}.$$

Answer: $\lambda \approx 7.27 \text{ Å}$.

Q3. Find the ratio of de Broglie wavelengths of a proton and an alpha particle accelerated through the same potential.

$$\lambda = h/\sqrt{2mqV} \Rightarrow \lambda \propto 1/\sqrt{mq}. \lambda_p/\lambda_\alpha = \sqrt{(m_\alpha q_\alpha)/(m_p q_p)} = \sqrt{(4 \times 2)} = \sqrt{8}.$$

Answer: $2\sqrt{2} \approx 2.83$.

Q4. The kinetic energy of an electron is doubled. How does its de Broglie wavelength change?

$$\lambda \propto 1/\sqrt{E} \Rightarrow \lambda' = \lambda/\sqrt{2}.$$

Answer: It decreases by a factor $\sqrt{2}$ (to about 70.7%).

Q5. The uncertainty in the position of an electron is 0.1 nm. Find the minimum uncertainty in its momentum.

$$\Delta p \geq \hbar/(2\Delta x) = 1.055 \times 10^{-34} / (2 \times 10^{-10}).$$

Answer: $\Delta p \geq 5.3 \times 10^{-25} \text{ kg m/s}$.

Q6. Find the lowest energy of a neutron ($m = 1.675 \times 10^{-27} \text{ kg}$) confined in a nucleus-sized box of 10^{-14} m .

$$E_1 = h^2/(8mL^2) = 4.39 \times 10^{-67} / (8 \times 1.675 \times 10^{-27} \times 10^{-28}) = 4.39 \times 10^{-67} / 1.34 \times 10^{-54} = 3.28 \times 10^{-13} \text{ J}.$$

Answer: $E_1 \approx 2.05 \text{ MeV}$.

Q7. An electron in a 1-D box has ground-state energy 2 eV. What is the energy of the third excited state?

$$\text{Third excited state is } n = 4: E_4 = 16E_1.$$

Answer: 32 eV.



Q8. How does the ground-state energy change if the width of the box is doubled?

$$E \propto 1/L^2 \Rightarrow E \text{ becomes } E/4.$$

Answer: It falls to one-quarter.

Q9. Show that $\psi = e^{ikx}$ is an eigenfunction of $\hat{p}$ and find the eigenvalue.

$$\hat{p}\psi = -i\hbar(ik)e^{ikx} = \hbar k e^{ikx}.$$

Answer: Eigenvalue $\hbar k = h/\lambda$.

Q10. For a particle in a box of length L in the $n = 2$ state, where is the probability of finding the particle maximum?

$|\psi_2|^2 \propto \sin^2(2\pi x/L)$ is maximum where $2\pi x/L = \pi/2, 3\pi/2 \Rightarrow x = L/4, 3L/4$. It is zero at $x = L/2$.

Answer: At $x = L/4$ and $x = 3L/4$.

4.22 MCQs: Module IV

1. The energy of a photon of frequency ν is

- (a) ν/h (b) $h\nu^2$ (c) $h\nu$ (d) h/ν

☐ (c) — Planck relation.

2. The momentum of a photon of wavelength λ is

- (a) h/λ (b) hc/λ (c) λ/h (d) $h\lambda$

☐ (a) — $p = h/\lambda$.

3. The photoelectric effect shows the

- (a) transverse nature of light (b) particle nature of light (c) wave nature of light (d) polarisation of light

☐ (b) — Photons.

4. Compton effect demonstrates

- (a) wave nature of X-rays (b) particle nature of X-rays (c) diffraction (d) interference

☐ (b) — Photon momentum.

5. The de Broglie wavelength of a particle is

- (a) h/E (b) h/p (c) hp (d) p/h

☐ (b) — $\lambda = h/mv$.

6. The de Broglie wavelength of an electron accelerated through V volts is

- (a) $1.227V \text{ \AA}$ (b) $12.27/\sqrt{V} \text{ \AA}$ (c) $12.27\sqrt{V} \text{ \AA}$ (d) $12.27/V \text{ \AA}$

☐ (b) — Shortcut.

7. The de Broglie wavelength is inversely proportional to

- (a) $\sqrt{E}$ (b) E (c) $1/E$ (d) E^2

☐ (a) — $\lambda = h/\sqrt{2mE}$.

8. Electron diffraction was first observed by

- (a) Einstein (b) Davisson and Germer (c) Planck (d) de Broglie

☐ (b) — 1927 experiment.

9. If the velocity of a particle is doubled, its de Broglie wavelength

- (a) stays same (b) becomes four times (c) is halved (d) doubles

☐ (c) — $\lambda \propto 1/v$.



10. A proton and an electron have the same kinetic energy. Which has the smaller de Broglie wavelength?

- (a) Cannot say (b) Both equal (c) Electron (d) Proton

□ (d) — $\lambda \propto 1/\sqrt{m}$.

11. Phase velocity is

- (a) $d\omega/dk$ (b) k/ω (c) ωk (d) ω/k

□ (d) — Definition.

12. Group velocity is

- (a) k/ω (b) ω/k (c) $dk/d\omega$ (d) $d\omega/dk$

□ (d) — Velocity of envelope.

13. For a de Broglie wave, the group velocity equals

- (a) the phase velocity (b) zero (c) c (d) the particle velocity

□ (d) — $v_g = p/m$.

14. For a relativistic particle, $v_p v_g$ equals

- (a) c (b) c^2 (c) 1 (d) v^2

□ (b) — $v_p = c^2/v$.

15. In a non-dispersive medium,

- (a) $v_g < v_p$ (b) $v_g = 0$ (c) $v_g > v_p$ (d) $v_g = v_p$

□ (d) — v_p independent of λ .

16. The phase velocity of matter waves can be

- (a) equal to c always (b) less than c only (c) equal to zero (d) greater than c

□ (d) — $v_p = c^2/v > c$.

17. The Heisenberg uncertainty relation is

- (a) $\Delta x \Delta p = 0$ (b) $\Delta x \Delta p \geq \hbar/2$ (c) $\Delta x/\Delta p \geq \hbar$ (d) $\Delta x + \Delta p \geq \hbar$

□ (b) — Position-momentum.

18. The physical meaning of $|\psi|^2$ is

- (a) probability density (b) energy (c) charge (d) momentum

□ (a) — Born interpretation.

19. A wave function must be

- (a) always real (b) infinite at some points (c) single-valued, finite and continuous (d) multivalued

□ (c) — Well-behaved conditions.

20. The normalisation condition is

- (a) $\int |\psi|^2 dx = 1$ (b) $\int \psi dx = 1$ (c) $\psi = 1$ (d) $\int |\psi|^2 dx = 0$

□ (a) — Total probability = 1.

21. The momentum operator is

- (a) $-i\hbar \partial/\partial x$ (b) $-\hbar^2 \partial^2/\partial x^2$ (c) $i\hbar \partial/\partial x$ (d) $\hbar \partial/\partial x$

□ (a) — $\hat{p} = -i\hbar \nabla$.

22. The energy operator is

- (a) $i\hbar \partial/\partial t$ (b) $\hbar \partial/\partial t$ (c) $-i\hbar \partial/\partial t$ (d) $\partial/\partial t$

□ (a) — $\hat{E} = i\hbar \partial/\partial t$.

23. If $\hat{A}\psi = a\psi$, a is called the

- (a) eigenvalue (b) operator (c) wave function (d) expectation value

□ (a) — Eigenvalue equation.

24. The expectation value of an observable A is

- (a) $\int |\hat{A}|^2 dx$ (b) $\int \psi^* \hat{A} \psi dx$ (c) $\int \hat{A} \psi dx$ (d) $\int \psi \hat{A} dx$

□ (b) — With normalised ψ .



- 25.** Eigenvalues of operators representing observables are
(a) real (b) imaginary (c) zero (d) complex
☐ (a) — Hermitian operators.
- 26.** The time-independent Schrödinger equation is
(a) $d^2\psi/dx^2 - E\psi = 0$ (b) $i\hbar \partial\psi/\partial t = V\psi$ (c) $d\psi/dx + (E - V)\psi = 0$ (d) $d^2\psi/dx^2 + (2m/\hbar^2)(E - V)\psi = 0$
☐ (d) — Standard form.
- 27.** Schrödinger's equation is
(a) independent of ψ (b) linear in ψ (c) quadratic in ψ (d) cubic in ψ
☐ (b) — Superposition holds.
- 28.** The energy of a particle in a 1-D box is proportional to
(a) $1/n$ (b) $\sqrt{n}$ (c) n^2 (d) n
☐ (c) — $E_n = n^2 h^2 / 8mL^2$.
- 29.** The ground-state energy of a particle in a box is
(a) infinite (b) $h^2/(4mL^2)$ (c) $h^2/(8mL^2)$ (d) zero
☐ (c) — $n = 1$.
- 30.** The zero-point energy exists because of
(a) gravity (b) Pauli's principle (c) the uncertainty principle (d) Coulomb's law
☐ (c) — Confined particle cannot be at rest.
- 31.** If the length of the box is halved, the ground-state energy becomes
(a) half (b) four times (c) double (d) one-fourth
☐ (b) — $E \propto 1/L^2$.
- 32.** The normalised wave function for a particle in a box is
(a) $\sqrt{1/L} \sin(n\pi x/L)$ (b) $\sqrt{2/L} \sin(n\pi x/L)$ (c) $(2/L) \sin(n\pi x/L)$ (d) $\sin(n\pi x/L)$
☐ (b) — $A = \sqrt{2/L}$.
- 33.** The number of nodes (excluding walls) in ψ_n of a particle in a box is
(a) $n - 1$ (b) n (c) $2n$ (d) $n + 1$
☐ (a) — Standing wave.
- 34.** The ratio $E_3 : E_1$ for a particle in a box is
(a) 6 (b) 27 (c) 3 (d) 9
☐ (d) — n^2 ratio.
- 35.** The value $n = 0$ is not allowed in the particle-in-a-box problem because
(a) the box collapses (b) it violates relativity (c) $\psi = 0$ everywhere (d) energy becomes infinite
☐ (c) — No particle.
- 36.** The probability of finding a particle in the ground state of a box is maximum at
(a) everywhere equal (b) the walls (c) the centre (d) $L/4$
☐ (c) — $\sin^2$ maximum at $L/2$.
- 37.** The allowed states of a particle in a box correspond to
(a) $L = \lambda/n$ (b) $L = 2n\lambda$ (c) $L = n\lambda/2$ (d) $L = n\lambda$
☐ (c) — Standing waves.
- 38.** The energy levels of a cubic box with $n_x^2 + n_y^2 + n_z^2 = 6$ are
(a) two-fold degenerate (b) non-degenerate (c) three-fold degenerate (d) six-fold degenerate
☐ (c) — (1,1,2) permutations.



- 39.** An electron cannot exist inside the nucleus because
(a) it is too heavy (b) it has no charge (c) of the uncertainty principle (d) of gravity
☐ (c) — Required energy too large.
- 40.** The energy of a photon of wavelength 1240 nm is
(a) 10 eV (b) 12.4 eV (c) 1 eV (d) 0.1 eV
☐ (c) — $12400/12400 \text{ \AA}$.

Module V: Lasers

LASER stands for **Light Amplification by Stimulated Emission of Radiation**. A laser produces an intense, highly directional, monochromatic and coherent beam of light. The theory was laid by Einstein (1917); the first working laser (ruby) was built by T.H. Maiman in 1960. Today lasers are used in optical fibre communication, barcode scanners, surgery, material cutting and welding, printers, holography and measurement.

5.1 Interaction of radiation with matter

Consider two energy levels of an atom, E_1 (lower) and E_2 (upper). A photon of energy $h\nu = E_2 - E_1$ can interact with the atom in three ways.

(1) Induced (stimulated) absorption. An atom in E_1 absorbs an incident photon and jumps to E_2 . Rate of absorption = $B_{12} \rho(\nu) N_1$, where N_1 is the number of atoms per unit volume in E_1 , $\rho(\nu)$ the energy density of radiation, and B_{12} the Einstein coefficient of absorption.

(2) Spontaneous emission. An atom in E_2 returns to E_1 on its own, after an average lifetime (about 10^{-8} s), emitting a photon of energy $h\nu$. The photons are emitted randomly in all directions with random phases, so the light is **incoherent** (e.g. ordinary lamps). Rate = $A_{21} N_2$, independent of the radiation present.

(3) Stimulated emission. An incident photon of energy $h\nu$ triggers an excited atom in E_2 to drop to E_1 , emitting a second photon. The emitted photon is **identical** to the incident photon: same frequency, phase, direction and polarisation. One photon in, two photons out: this is **light amplification**. Rate = $B_{21} \rho(\nu) N_2$.

Spontaneous emission	Stimulated emission
Occurs without any external photon	Requires an incident photon of the right energy
Photons emitted randomly in all directions	Emitted photon travels in the same direction as the incident photon
Incoherent (random phases)	Coherent (same phase as incident photon)
Not monochromatic (broad spectrum)	Highly monochromatic
Rate $\propto N_2$ only	Rate $\propto N_2 \rho(\nu)$
Example: sodium lamp, LED, sunlight	Example: laser

5.2 Characteristics of laser light

- **Monochromaticity:** laser light has an extremely narrow spectral width ($\Delta\lambda$ as small as 10^{-6} nm for gas lasers), compared with about 100 nm for white light.
- **Directionality:** the beam spreads very little. Angular divergence $\theta \approx \lambda/d$ (about 10^{-3} – 10^{-4} rad). A laser beam expanded by a telescope and sent to the Moon spreads to only a few kilometres.
- **Coherence:** all waves are in phase. **Temporal coherence** (along the beam) is measured by the coherence time $\tau_c \approx 1/\Delta\nu$ and coherence length $L_c = c\tau_c$ (up to kilometres for a stabilised He-Ne laser, but a few μm for white light). **Spatial coherence** means points across the beam are in phase.
- **High intensity (brightness):** because all the energy is concentrated in a narrow beam, even a 1 mW laser is far brighter per unit area than a 100 W bulb. Focused high-power lasers can cut steel.

5.3 Einstein's coefficients and the relation between them

At thermal equilibrium the rate of upward transitions equals the rate of downward transitions:

$$B_{12} \rho(\nu) N_1 = A_{21} N_2 + B_{21} \rho(\nu) N_2$$

Solving for the radiation energy density:

$$\rho(\nu) = A_{21} / [B_{12}(N_1/N_2) - B_{21}]$$

At equilibrium the populations follow the Boltzmann distribution:

$$N_2/N_1 = e^{-(E_2 - E_1)/k_B T} = e^{-h\nu/k_B T}$$

So $\rho(\nu) = A_{21} / [B_{12}e^{h\nu/k_B T} - B_{21}] = (A_{21}/B_{21}) \times 1/[(B_{12}/B_{21})e^{h\nu/k_B T} - 1]$.

Compare with **Planck's radiation law**: $\rho(\nu) = (8\pi h\nu^3/c^3) \times 1/(e^{h\nu/k_B T} - 1)$. For the two expressions to agree at all temperatures:

$$B_{12} = B_{21} \quad A_{21}/B_{21} = 8\pi h\nu^3/c^3$$

Consequences.

- Since $B_{12} = B_{21}$, the probability of stimulated emission equals that of absorption. Whether light is amplified or absorbed depends only on which level has more atoms.
- The ratio of spontaneous to stimulated emission rates is $R = A_{21}/(B_{21}\rho(\nu)) = e^{h\nu/k_B T} - 1$. For visible light at room temperature ($h\nu \approx 2$ eV, $k_B T \approx 0.025$ eV), $R \approx e^{80}$, so spontaneous emission completely dominates in ordinary sources.
- $A_{21}/B_{21} \propto \nu^3$: spontaneous emission becomes much stronger at higher frequencies, which is why X-ray lasers are very difficult to make.

5.4 Population inversion

In thermal equilibrium, $N_2 < N_1$ (Boltzmann distribution), so absorption dominates and light is attenuated. For amplification by stimulated emission we need

$$N_2 > N_1 \text{ (population inversion)}$$

This is a non-equilibrium state (it would correspond to a "negative temperature"). It cannot be achieved in a two-level system by optical pumping, because once $N_2 = N_1$, absorption and stimulated emission balance ($B_{12} = B_{21}$) and the medium becomes transparent. Three- or four-level systems with a **metastable state** are therefore needed.

Metastable state: an excited state with an unusually long lifetime (about 10^{-3} s, compared with 10^{-8} s for ordinary excited states). Atoms accumulate there, making population inversion possible.

5.5 Pumping

Pumping is the process of supplying energy to the active medium to raise atoms to higher levels and create population inversion.

- **Optical pumping**: using a flash lamp or another laser (Ruby laser, Nd:YAG laser).
- **Electrical discharge pumping**: electrons in a gas discharge excite atoms by collision (He-Ne laser, CO₂ laser, argon-ion laser).

- **Direct conversion (injection pumping):** forward current in a p-n junction (semiconductor diode laser).
- **Chemical pumping:** energy from a chemical reaction (HF laser).
- **Inelastic atom-atom collisions:** energy transfer from one gas to another (He to Ne in the He-Ne laser).

5.6 Components of a laser

- **Active (gain) medium:** the material in which population inversion is produced and stimulated emission occurs (ruby crystal, He-Ne gas mixture, semiconductor).
- **Pumping source (energy source):** supplies energy to achieve population inversion.
- **Optical resonator (cavity):** two parallel mirrors at the ends of the active medium, one fully reflecting ($\approx 100\%$) and one partially reflecting ($\approx 95\text{--}99\%$). Photons travelling along the axis bounce back and forth, stimulating more emission at each pass; this builds up an intense, coherent, directional beam. The partially reflecting mirror lets part of it out as the laser output. Photons travelling in other directions leave the cavity and are lost. The cavity supports only wavelengths that satisfy $2L = m\lambda$ (longitudinal modes).
- **Threshold condition:** the gain per round trip must exceed the losses (mirror transmission, scattering, absorption). Laser action starts only above a threshold pump power.

5.7 Lasing action (how the beam builds up)

- **Step 1, pumping:** the pump raises atoms from the ground state to higher levels.
- **Step 2, fast decay to the metastable level:** atoms drop quickly (non-radiatively) to the metastable upper laser level, where they remain for a long time. The population there builds up until $N_{\text{upper}} > N_{\text{lower}}$: population inversion.
- **Step 3, seed photon:** a few atoms decay spontaneously, emitting photons in random directions.
- **Step 4, stimulated emission chain:** a photon travelling along the cavity axis stimulates an excited atom to emit an identical photon. These two stimulate two more, and so on: an avalanche of identical photons.
- **Step 5, optical feedback:** the mirrors reflect the photons back and forth through the medium many times, multiplying them further.
- **Step 6, output:** when the gain exceeds the losses, a steady, intense, coherent beam emerges through the partially reflecting mirror.

5.8 Three-level pumping scheme

Levels: E_1 (ground state, lower laser level), E_2 (metastable upper laser level), E_3 (pump band, short-lived).

- Atoms are pumped from E_1 to E_3 .
- They decay rapidly ($\approx 10^{-8}$ s) and non-radiatively to the metastable level E_2 .
- Laser transition: $E_2 \rightarrow E_1$.

Drawback. The lower laser level is the ground state, which is initially full. More than half of all the atoms must be pumped to E_2 to achieve inversion. This needs very intense pumping, so three-level lasers usually work in **pulsed** mode and are inefficient. Example: **Ruby laser**.

5.9 Four-level pumping scheme

Levels: E_0 (ground), E_3 (pump band), E_2 (metastable upper laser level), E_1 (lower laser level, well above E_0 , short-lived).

- Atoms are pumped $E_0 \rightarrow E_3$, then decay rapidly to E_2 .
- Laser transition: $E_2 \rightarrow E_1$.
- Atoms in E_1 decay quickly to E_0 , so E_1 stays almost empty.

Advantage. Since E_1 is nearly empty, population inversion between E_2 and E_1 is achieved even when only a small fraction of atoms is excited. The threshold is much lower, and **continuous-wave (CW)** operation is easy. Examples: **He-Ne laser**, Nd:YAG laser, CO_2 laser.

Three-level laser	Four-level laser
Lower laser level is the ground state	Lower laser level is above the ground state
More than 50% of atoms must be excited for inversion	Inversion with a small fraction excited
High threshold, low efficiency	Low threshold, higher efficiency
Usually pulsed output	Continuous output possible
Example: Ruby laser	Examples: He-Ne, Nd:YAG, CO_2

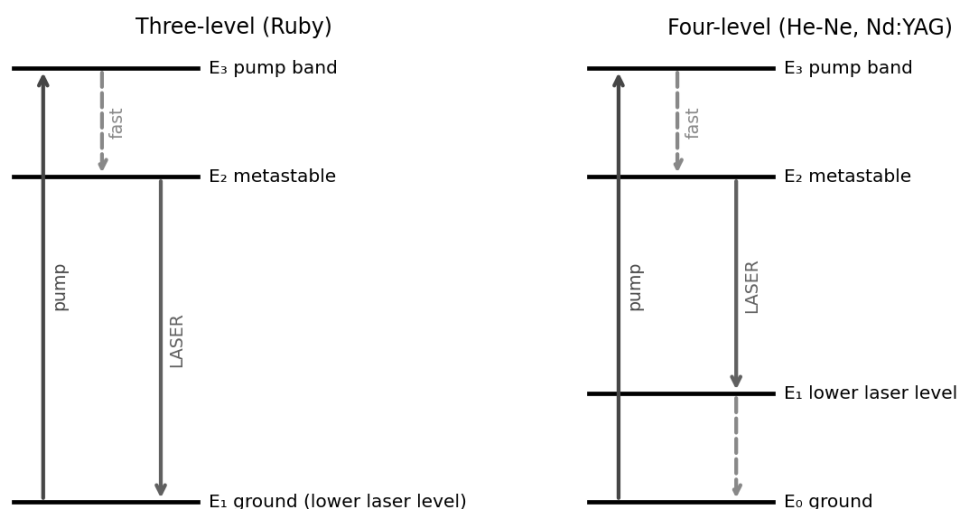


Figure 5.1: Three-level and four-level pumping schemes.

5.10 Ruby laser (first laser, Maiman 1960)

Construction.

- **Active medium:** a cylindrical rod of ruby, which is aluminium oxide (Al_2O_3) doped with about 0.05% chromium ions (Cr^{3+}). The Cr^{3+} ions are the active centres; Al_2O_3 is the host. Typical rod: 2-20 cm long, 0.5-2 cm diameter.
- **Resonator:** the two end faces are ground flat and parallel and silvered: one fully, one partially reflecting.
- **Pump:** a helical xenon flash lamp wound around the rod, fired by a high-voltage capacitor bank. A cooling system removes the heat.

Working (three-level system).

- Cr^{3+} ions absorb green ($\approx 5500 \text{ \AA}$) and blue ($\approx 4000 \text{ \AA}$) light from the flash lamp and go from the ground state E_1 to the broad pump bands E_3 .
- They decay non-radiatively in about 10^{-8} s to the metastable level E_2 (lifetime $\approx 3 \text{ ms}$), giving up the energy to the crystal lattice as heat.
- When population inversion between E_2 and E_1 is reached, a spontaneously emitted photon starts stimulated emission; the chain reaction builds up between the mirrors.
- Output: an intense pulse of **red light at $6943 \text{ \AA}$** .

Characteristics: three-level, pulsed output (pulses of about 1 ms), power up to several kilowatts to megawatts in the pulse, efficiency very low ($\approx 0.1\%$).

Drawbacks: very high pumping power needed; pulsed only; the rod heats up; low efficiency; laser output is not very uniform (spiky).

Uses: pulsed holography, range finding, drilling holes in diamonds and hard materials, early tattoo and pigment removal.

5.11 Helium-Neon (He-Ne) laser (Javan, 1961)

Construction.

- **Active medium:** a gas mixture of helium and neon in the ratio about $10 : 1$ at low pressure ($\approx 1 \text{ mm of Hg}$) in a long, narrow quartz discharge tube (length $\approx 50\text{--}80 \text{ cm}$, diameter $\approx 1 \text{ cm}$). Neon atoms produce the laser light; helium atoms help to pump them.
- **Pump:** an electric discharge produced by a high DC voltage ($\approx 1\text{--}2 \text{ kV}$) applied between electrodes.
- **Resonator:** two mirrors (often concave) at the ends, one fully and one partially reflecting. The tube ends are often closed with Brewster windows so the output is plane polarised.

Working (four-level system, pumping by atom-atom collisions).

- The discharge accelerates electrons, which collide with helium atoms and excite them to the metastable levels F_2 (19.81 eV) and F_3 (20.61 eV). These He levels are long-lived because transitions to the ground state are forbidden.
- The excited He levels have almost the same energy as the neon levels E_4 (19.83 eV , the $3s$ level) and E_6 (20.66 eV , the $5s$ level). Excited He atoms collide with ground-state Ne atoms and **transfer their energy** by resonant collision: $\text{He}^* + \text{Ne} \rightarrow \text{He} + \text{Ne}^*$. The Ne upper levels become heavily populated.
- The Ne lower levels E_3 (18.70 eV , $3p$) and E_5 (20.30 eV , $4p$) are nearly empty, so population inversion is established.
- Laser transitions in neon: $E_6 \rightarrow E_5$: $3.39 \text{ \mu m}$ (IR); $E_6 \rightarrow E_3$: **$6328 \text{ \AA}$ (red, the familiar visible line)**; $E_4 \rightarrow E_3$: $1.15 \text{ \mu m}$ (IR).
- From E_3 , atoms decay by spontaneous emission to the $3s$ level ($\approx 6000 \text{ \AA}$), then return to the ground state mainly by collisions with the tube walls. This is why a narrow tube improves performance.

Characteristics: four-level, **continuous** output, power $0.5\text{--}50 \text{ mW}$, highly monochromatic and coherent, good directionality, efficiency about 0.1% .

Advantages over ruby: continuous operation, better monochromaticity and coherence, no cooling needed, simple and cheap to run.

Uses: barcode scanners, laboratory alignment and interferometry, holography, laser printers (earlier), pointing, metrology, optical experiments.

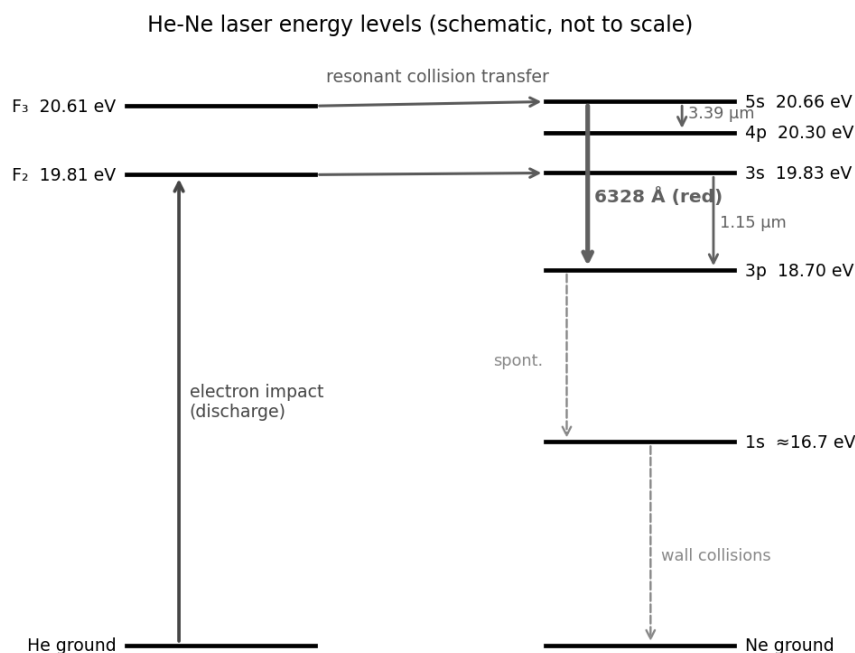


Figure 5.2: Energy levels of the He-Ne laser showing collision transfer from helium to neon and the three laser lines.

5.12 Applications of lasers (summary)

- **Communication:** optical fibre systems (semiconductor lasers at 1.3 and 1.55 μm).
- **Industry:** cutting, welding, drilling and marking (CO₂ and Nd:YAG lasers).
- **Medicine:** eye surgery (LASIK), removal of tumours, bloodless surgery, dentistry, cosmetic treatments.
- **Science and measurement:** interferometry, holography, spectroscopy, LIDAR, measuring the Earth-Moon distance, gravitational-wave detection.
- **Consumer and IT:** CD/DVD/Blu-ray players, laser printers, barcode readers, laser pointers.
- **Defence:** range finders, target designators.

5.13 Common mistakes to avoid in Module V

- Saying stimulated emission produces a photon "of any frequency". It produces a photon identical to the incident one.
- Claiming population inversion can be produced in a two-level system by optical pumping. It cannot.
- Writing $A_{21}/B_{21} = 8\pi h\nu^3/c^2$ (wrong power of c). It is c^3 .
- Giving the ruby wavelength as 6328 Å or the He-Ne wavelength as 6943 Å. Ruby: 6943 Å; He-Ne: 6328 Å.
- Stating that neon is excited directly by the discharge as the main mechanism. In the He-Ne laser the main pumping of neon is by collisions with metastable helium atoms.

5.14 Planck's law from Einstein's coefficients (full derivation)

At equilibrium: $B_{12}\rho N_1 = A_{21}N_2 + B_{21}\rho N_2$. Rearranging: $\rho(B_{12}N_1 - B_{21}N_2) = A_{21}N_2$, so

$$\rho(\nu) = A_{21}N_2/(B_{12}N_1 - B_{21}N_2) = A_{21}/(B_{12}(N_1/N_2) - B_{21}).$$

$$\text{Using } N_1/N_2 = e^{h\nu/k_B T}: \rho(\nu) = A_{21}/(B_{12}e^{h\nu/k_B T} - B_{21}).$$

- As $T \rightarrow \infty$, $\rho \rightarrow \infty$ must hold (the radiation density grows without limit). Then $e^{h\nu/k_B T} \rightarrow 1$, and the denominator $B_{12} - B_{21}$ must be 0. Hence $B_{12} = B_{21} = B$.
- So $\rho(\nu) = (A_{21}/B)/(e^{h\nu/k_B T} - 1)$. Comparing with Planck's law $\rho(\nu) = (8\pi h\nu^3/c^3)/(e^{h\nu/k_B T} - 1)$ gives $A_{21}/B = 8\pi h\nu^3/c^3$.

If the levels are degenerate with degeneracies g_1, g_2 , the relation becomes $g_1 B_{12} = g_2 B_{21}$.

5.15 Gain, threshold and the role of the resonator

When light of intensity I passes through a medium with populations N_1, N_2 , the change in intensity over a small length dz is proportional to $(N_2 - N_1)$:

$$dI/dz = \sigma(N_2 - N_1) I \Rightarrow I(z) = I_0 e^{\sigma(N_2 - N_1)z}$$

(σ is the stimulated-emission cross-section.) If $N_2 < N_1$, the exponent is negative and the light is absorbed (Beer-Lambert law). If $N_2 > N_1$ (population inversion), the medium has **gain** and the light is amplified.

Threshold condition. In a cavity of length L with mirror reflectivities R_1, R_2 , after one round trip the intensity becomes $I R_1 R_2 e^{2gL}$ (g = gain coefficient, ignoring other losses). Oscillation builds up when $R_1 R_2 e^{2gL} \geq 1$, i.e.

$$g_{th} = (1/2L) \ln(1/(R_1 R_2)) \text{ (+ internal loss coefficient, if any)}$$

Longitudinal modes. Standing waves require $L = m\lambda/2$, so allowed frequencies are $\nu_m = mc/(2L)$, separated by $\Delta\nu = c/(2L)$. Only the modes that fall within the gain bandwidth of the medium and exceed threshold oscillate.

5.16 Comparison of Ruby and He-Ne lasers

Feature	Ruby laser	He-Ne laser
Type	Solid-state	Gas
Active medium	Al_2O_3 doped with 0.05% Cr^{3+}	He : Ne $\approx 10 : 1$ at low pressure
Lasing species	Cr^{3+} ions	Ne atoms
Level scheme	Three-level	Four-level
Pumping	Optical (xenon flash lamp)	Electrical discharge + He-Ne collisions
Output	Pulsed	Continuous (CW)
Wavelength	6943 Å (red)	6328 Å (red); also 1.15 μm , 3.39 μm
Power	High peak power (kW-MW)	Low (0.5-50 mW)
Efficiency	$\approx 0.1\%$	$\approx 0.1\%$
Monochromaticity / coherence	Good	Excellent
Cooling	Required	Not needed
Uses	Drilling, range finding, pulsed holography	Alignment, barcode scanners, interferometry, holography

5.17 Other important lasers (for general awareness)

- **Nd:YAG laser:** neodymium ions in yttrium aluminium garnet; four-level; 1.064 μm (IR); pulsed or CW; used in material processing, surgery and range finders.
- **CO₂ laser:** molecular gas laser using vibrational levels of CO₂ (with N₂ and He); 10.6 μm (far IR); very high CW power and efficiency ($\approx 30\%$); used for cutting and welding.
- **Semiconductor diode laser:** population inversion at a heavily doped p-n junction by forward current (injection pumping); the cleaved crystal faces act as mirrors; wavelengths from blue to IR depending on the band gap ($\lambda = hc/E_g$); tiny, efficient and cheap; used in optical fibre communication, CD/DVD players and laser pointers.
- **Dye laser:** organic dye solution; tunable over a wide wavelength range; used in spectroscopy.
- **Fibre laser:** rare-earth-doped optical fibre; high power and excellent beam quality; used in industrial cutting.

5.18 Laser safety

Laser light is concentrated, so even a few milliwatts focused by the eye's lens onto the retina can cause permanent damage. Lasers are classified from Class 1 (safe) to Class 4 (dangerous to eyes and skin, fire hazard). Never look into the beam or its specular reflection; use appropriate goggles; keep beams at below eye level in laboratories.

5.19 Solved numericals: Module V

Problem 1. Find the frequency and photon energy (in eV) of the ruby laser line at 6943 Å.

$$\nu = c/\lambda = 3 \times 10^8 / 6943 \times 10^{-10} = 4.32 \times 10^{14} \text{ Hz.}$$

$$E = h\nu = 6.626 \times 10^{-34} \times 4.32 \times 10^{14} = 2.86 \times 10^{-19} \text{ J} = 1.79 \text{ eV.}$$

Answer: $\nu \approx 4.32 \times 10^{14} \text{ Hz}$; $E \approx 1.79 \text{ eV}$.

Problem 2. Find the ratio of populations N_2/N_1 of the two ruby laser levels at 300 K in thermal equilibrium. Comment.

$$N_2/N_1 = e^{-h\nu/k_B T}. \quad h\nu/k_B T = 2.86 \times 10^{-19} / (1.38 \times 10^{-23} \times 300) = 69.1.$$

$$N_2/N_1 = e^{-69.1} \approx 9 \times 10^{-31}.$$

Answer: $\approx 10^{-30}$. The upper level is practically empty at room temperature, so population inversion must be created artificially by pumping.

Problem 3. A He-Ne laser emits 1 mW at 6328 Å. How many photons does it emit per second?

$$\text{Energy per photon } E = hc/\lambda = 6.626 \times 10^{-34} \times 3 \times 10^8 / 6.328 \times 10^{-7} = 3.14 \times 10^{-19} \text{ J (1.96 eV).}$$

$$n = P/E = 10^{-3} / 3.14 \times 10^{-19}.$$

Answer: $n \approx 3.2 \times 10^{15}$ photons per second.

Problem 4. Find the ratio of spontaneous to stimulated emission rates for the 6328 Å transition at 300 K in thermal equilibrium.

$$R = e^{h\nu/k_B T} - 1. \quad h\nu/k_B T = 3.14 \times 10^{-19} / (4.14 \times 10^{-21}) = 75.8.$$

$$R \approx e^{75.8} \approx 9 \times 10^{32}.$$

Answer: Spontaneous emission is about 10^{33} times more likely, which is why ordinary sources are incoherent.

Problem 5. At what temperature are the rates of spontaneous and stimulated emission equal for $\lambda = 6000 \text{ \AA}$?

$$\text{Equal rates} \Rightarrow e^{h\nu/k_B T} - 1 = 1 \Rightarrow h\nu/k_B T = \ln 2.$$

$$T = hc/(\lambda k_B \ln 2) = 6.626 \times 10^{-34} \times 3 \times 10^8 / (6 \times 10^{-7} \times 1.38 \times 10^{-23} \times 0.693).$$

Answer: $T \approx 3.46 \times 10^4 \text{ K}$.

Problem 6. Calculate A_{21}/B_{21} for a transition at $\lambda = 5000 \text{ \AA}$.

$$A/B = 8\pi h\nu^3/c^3 = 8\pi h/\lambda^3 = 8 \times 3.1416 \times 6.626 \times 10^{-34} / (5 \times 10^{-7})^3 = 1.665 \times 10^{-32} / 1.25 \times 10^{-19}.$$

Answer: $A_{21}/B_{21} \approx 1.33 \times 10^{-13} \text{ J s m}^{-3}$.

Problem 7. A ruby laser emits a pulse of energy 1 J. How many Cr^{3+} ions must have made the laser transition?

$$\text{Each photon: } E = hc/\lambda = 2.86 \times 10^{-19} \text{ J}.$$

$$n = 1/2.86 \times 10^{-19}.$$

Answer: About 3.5×10^{18} ions.

Problem 8. A laser pulse of energy 0.5 J lasts 1 ns. Find its peak power.

$$P = E/t = 0.5/10^{-9}.$$

Answer: $P = 5 \times 10^8 \text{ W}$ (500 MW).

Problem 9. A 5 mW laser beam has a diameter of 2 mm. Find its intensity. Compare with sunlight (1 kW/m^2).

$$\text{Area} = \pi(10^{-3})^2 = 3.14 \times 10^{-6} \text{ m}^2. I = 5 \times 10^{-3} / 3.14 \times 10^{-6} = 1592 \text{ W/m}^2.$$

Answer: $I \approx 1.6 \text{ kW/m}^2$, more than direct sunlight. Never look into a laser beam.

Problem 10. A laser beam of wavelength $6328 \text{ \AA}$ has an aperture of 5 mm. Find its angular divergence and the spot diameter at a distance of 1 km.

$$\theta \approx \lambda/d = 6.328 \times 10^{-7} / 5 \times 10^{-3} = 1.27 \times 10^{-4} \text{ rad}.$$

$$\text{Taking } \theta \text{ as the half-angle of spread, spot diameter} \approx 2L\theta = 2 \times 1000 \times 1.27 \times 10^{-4} \approx 0.25 \text{ m (the initial 5 mm is negligible).}$$

Answer: $\theta \approx 1.27 \times 10^{-4} \text{ rad}$; spot $\approx 25 \text{ cm}$ across at 1 km.

Problem 11. The line width of a laser is $\Delta\lambda = 10^{-14} \text{ m}$ at $\lambda = 6328 \text{ \AA}$. Find the coherence length and coherence time.

$$L_c = \lambda^2/\Delta\lambda = (6.328 \times 10^{-7})^2 / 10^{-14} = 4.0 \times 10^{-13} / 10^{-14} = 40 \text{ m}.$$

$$\tau_c = L_c/c = 40/3 \times 10^8 = 1.33 \times 10^{-7} \text{ s}.$$

Answer: $L_c \approx 40$ m; $\tau_c \approx 0.13$ μ s.

Problem 12. The coherence time of a source is 10^{-10} s. Find the coherence length and the spectral width $\Delta\nu$.

$$L_c = c\tau_c = 3 \times 10^8 \times 10^{-10} = 0.03 \text{ m.}$$

$$\Delta\nu \approx 1/\tau_c = 10^{10} \text{ Hz.}$$

Answer: $L_c = 3$ cm; $\Delta\nu \approx 10$ GHz.

Problem 13. A semiconductor laser emits at 1.55 μ m. Find the band gap of the material in eV.

$$E_g \approx hc/\lambda = 1.24/1.55 \text{ eV (using } hc = 1.24 \text{ eV } \mu\text{m)}.$$

Answer: $E_g \approx 0.80$ eV (InGaAsP, used in optical fibre communication).

Problem 14. In a He-Ne laser, the energies of the Ne levels involved in the 6328 Å transition are 20.66 eV and 18.70 eV. Verify the wavelength.

$$\Delta E = 20.66 - 18.70 = 1.96 \text{ eV. } \lambda = 12400/1.96 \text{ Å.}$$

Answer: $\lambda \approx 6327$ Å.

Problem 15. The metastable level of a three-level laser has a lifetime of 3 ms, while ordinary excited states live 10 ns. By what factor is the metastable state longer-lived?

$$3 \times 10^{-3}/10^{-8} = 3 \times 10^5.$$

Answer: About 300,000 times longer, which lets atoms accumulate for population inversion.

Problem 16. For a two-level system at 1000 K with energy gap 1 eV, find N_2/N_1 .

$$E/k_B T = 1.602 \times 10^{-19}/(1.38 \times 10^{-23} \times 1000) = 11.6.$$

Answer: $N_2/N_1 = e^{-11.6} \approx 9.1 \times 10^{-6}$.

Problem 17. The mirror separation in a He-Ne laser cavity is 30 cm. Find the frequency separation between adjacent longitudinal modes.

$$\text{Allowed modes satisfy } 2L = m\lambda \Rightarrow \nu_m = mc/2L. \text{ Spacing } \Delta\nu = c/2L = 3 \times 10^8/0.6.$$

Answer: $\Delta\nu = 5 \times 10^8$ Hz = 500 MHz.

5.20 More solved numericals: Module V

Problem 1. A laser cavity has mirrors of reflectivity 100% and 98% and length 50 cm. Find the threshold gain coefficient (ignore other losses).

$$g_{th} = (1/2L) \ln(1/(R_1 R_2)) = (1/1.0) \ln(1/0.98) = 0.0202 \text{ m}^{-1}.$$

Answer: $g_{th} \approx 0.02$ m^{-1} (2% per metre).

Problem 2. Light passes through a 10 cm long amplifying medium with gain coefficient 0.05 cm^{-1} . By what factor is its intensity increased?

$$I/I_0 = e^{gL} = e^{0.05 \times 10} = e^{0.5}.$$

Answer: About 1.65 times.

Problem 3. Calculate the number of longitudinal modes in a He-Ne laser ($L = 30 \text{ cm}$) if the gain bandwidth is 1.5 GHz.

$$\text{Mode spacing } \Delta\nu = c/2L = 3 \times 10^8/0.6 = 500 \text{ MHz. Number of modes } \approx 1500/500 = 3.$$

Answer: About 3 modes.

Problem 4. A pulsed ruby laser emits 10^{19} photons in a 1 ms pulse. Find the pulse energy and average power during the pulse.

$$\text{Energy per photon} = 2.86 \times 10^{-19} \text{ J. Total } E = 10^{19} \times 2.86 \times 10^{-19} = 2.86 \text{ J.}$$

$$P = E/t = 2.86/10^{-3} = 2860 \text{ W.}$$

Answer: $E \approx 2.86 \text{ J}$; $P \approx 2.86 \text{ kW}$.

Problem 5. A semiconductor laser has a band gap of 1.42 eV (GaAs). Find the emission wavelength.

$$\lambda = 1.24/1.42 \text{ } \mu\text{m}.$$

Answer: $\lambda \approx 0.873 \text{ } \mu\text{m}$ (873 nm, near infrared).

Problem 6. What band gap is needed for a blue laser diode emitting at 405 nm?

$$E_g = 1240/405 \text{ eV.}$$

Answer: $E_g \approx 3.06 \text{ eV}$ (GaN-based, used in Blu-ray).

Problem 7. At 300 K, find N_2/N_1 for levels separated by (a) 0.01 eV, (b) 2 eV.

$$k_B T = 0.02585 \text{ eV. (a) } e^{-0.01/0.02585} = e^{-0.387} = 0.68. \text{ (b) } e^{-2/0.02585} = e^{-77.4} \approx 2.6 \times 10^{-34}.$$

Answer: (a) 0.68 (upper level well populated); (b) $\approx 10^{-34}$ (upper level practically empty).

Problem 8. The coherence length of a sodium lamp ($5893 \text{ \AA}$) is about 3 cm. Find its spectral line width in $\text{\AA}$.

$$\Delta\lambda = \lambda^2/L_c = (5.893 \times 10^{-7})^2/0.03 = 3.473 \times 10^{-13}/0.03 = 1.16 \times 10^{-11} \text{ m.}$$

Answer: $\Delta\lambda \approx 0.12 \text{ \AA}$.

Problem 9. A He-Ne laser of 5 mW is focused to a spot of diameter 10 μm . Find the intensity at the spot.

$$\text{Area} = \pi(5 \times 10^{-6})^2 = 7.85 \times 10^{-11} \text{ m}^2. I = 5 \times 10^{-3}/7.85 \times 10^{-11}.$$

Answer: $I \approx 6.4 \times 10^7 \text{ W/m}^2$ (about 64,000 times stronger than sunlight). This is why lasers are dangerous to the eye.

Problem 10. How long must a ruby laser be pumped to accumulate enough Cr^{3+} ions in the metastable state if the lifetime is 3 ms? What happens if pumping is slower?

The metastable state must be filled faster than it empties, i.e. within a time comparable to its lifetime ($\approx 3 \text{ ms}$). If pumping is slower, atoms decay spontaneously before inversion is reached.

Answer: Pumping must be intense and fast (within a few ms), which is why a powerful flash lamp is used.

Problem 11. For a transition at $1 \mu\text{m}$, find the temperature at which the Boltzmann population ratio N_2/N_1 equals 0.1.

$$h\nu = 1.24 \text{ eV}. e^{-1.24/k_B T} = 0.1 \Rightarrow 1.24/k_B T = \ln 10 = 2.303 \Rightarrow k_B T = 0.538 \text{ eV}.$$

Answer: $T = 0.538/8.617 \times 10^{-5} \approx 6250 \text{ K}$. Even at such temperatures $N_2 < N_1$; thermal heating can never produce inversion.

5.21 Practice problems with solutions: Module V

Q1. Find the energy in eV of a photon from a CO_2 laser ($\lambda = 10.6 \mu\text{m}$).

$$E = 1.24/10.6 \text{ eV}.$$

Answer: $E \approx 0.117 \text{ eV}$.

Q2. A 2 mW He-Ne laser ($6328 \text{ \AA}$) runs for 1 minute. How many photons are emitted?

$$\text{Photons per second} = 2 \times 10^{-3} / 3.14 \times 10^{-19} = 6.37 \times 10^{15}. \text{ In } 60 \text{ s: } 3.8 \times 10^{17}.$$

Answer: About 3.8×10^{17} photons.

Q3. The ratio N_2/N_1 for two levels is $1/e$ at a certain temperature. If the energy gap is 0.1 eV, find the temperature.

$$e^{-\Delta E/k_B T} = e^{-1} \Rightarrow k_B T = \Delta E \Rightarrow T = 0.1 \times 1.602 \times 10^{-19} / 1.38 \times 10^{-23}.$$

Answer: $T \approx 1161 \text{ K}$.

Q4. A laser has a coherence length of 1 km. Find its spectral line width $\Delta\nu$.

$$\Delta\nu = c/L_c = 3 \times 10^8/10^3.$$

Answer: $\Delta\nu = 3 \times 10^5 \text{ Hz} = 300 \text{ kHz}$.

Q5. Find the peak power of a Nd:YAG laser pulse of 100 mJ energy and 10 ns duration.

$$P = 0.1/10^{-8}.$$

Answer: $10^7 \text{ W} = 10 \text{ MW}$.

Q6. Why is population inversion impossible in a two-level system with optical pumping?

$B_{12} = B_{21}$, so as N_2 approaches N_1 , the pump photons cause as many stimulated emissions as absorptions. The populations can at most become equal (the medium becomes transparent) but N_2 cannot exceed N_1 .

Answer: Because absorption and stimulated emission have equal probability.

Q7. Explain why the He-Ne laser tube is made narrow.

Neon atoms in the 1s (lower) levels return to the ground state mainly by colliding with the tube walls. A narrow tube increases wall collisions, emptying the lower levels quickly and maintaining population inversion.

Answer: To depopulate the lower laser levels by wall collisions.

Q8. Compare the photon energies of the ruby (6943 Å) and He-Ne (6328 Å) lasers.

$E_{\text{ruby}} = 12400/6943 = 1.786 \text{ eV}$; $E_{\text{He-Ne}} = 12400/6328 = 1.960 \text{ eV}$.

Answer: He-Ne photons carry about 10% more energy.

5.22 MCQs: Module V**1. LASER stands for**

- (a) Light Absorption by Stimulated Emission of Radiation (b) Light Amplification by Spontaneous Emission of Radiation (c) Light Amplification by Stimulated Emission of Radiation (d) Low Amplitude Stimulated Emission of Radiation
☐ (c) — Definition.

2. The first laser was built by

- (a) Townes (b) Einstein (c) Javan (d) Maiman
☐ (d) — Ruby laser, 1960.

3. The theoretical basis of the laser (stimulated emission) was given by

- (a) Einstein (b) Bohr (c) Maxwell (d) Planck
☐ (a) — 1917.

4. In stimulated emission, the emitted photon has the same

- (a) frequency only (b) energy as the atom (c) frequency, phase, direction and polarisation as the incident photon (d) phase only
☐ (c) — Identical photons.

5. Spontaneous emission produces light that is

- (a) incoherent (b) coherent (c) polarised (d) monochromatic
☐ (a) — Random phases and directions.

6. The relation between Einstein's B coefficients is

- (a) $B_{12} = 0$ (b) $B_{12} = B_{21}$ (c) $B_{12} = 2B_{21}$ (d) $B_{21} = 0$
☐ (b) — From Planck's law.

7. A_{21}/B_{21} equals

- (a) $h\nu/k_B T$ (b) $8\pi h\nu/c$ (c) $8\pi h\nu^2/c^2$ (d) $8\pi h\nu^3/c^3$
☐ (d) — Einstein relation.



8. Population inversion means

- (a) $N_2 > N_1$ (b) $N_1 > N_2$ (c) $N_1 = 0$ (d) $N_1 = N_2$
☐ (a) — More atoms in the upper level.

9. At thermal equilibrium,

- (a) $N_1 = 0$ (b) $N_2 < N_1$ (c) $N_2 = N_1$ (d) $N_2 > N_1$
☐ (b) — Boltzmann distribution.

10. The lifetime of a metastable state is about

- (a) 10^{-3} s (b) 10^{-15} s (c) 10^{-8} s (d) 1 hour
☐ (a) — Long compared with ordinary states.

11. The process of supplying energy to achieve population inversion is called

- (a) cooling (b) doping (c) pumping (d) lasing
☐ (c) — Definition.

12. The ruby laser uses

- (a) injection pumping (b) electrical pumping (c) optical pumping (d) chemical pumping
☐ (c) — Xenon flash lamp.

13. The He-Ne laser uses

- (a) chemical pumping (b) optical pumping (c) thermal pumping (d) electrical discharge pumping
☐ (d) — DC discharge.

14. The active centres in a ruby laser are

- (a) Al^{3+} ions (b) Cr^{3+} ions (c) Ne atoms (d) He atoms
☐ (b) — Chromium dopant.

15. The wavelength of the ruby laser is

- (a) $10.6 \mu\text{m}$ (b) $5890 \text{ \AA}$ (c) $6328 \text{ \AA}$ (d) $6943 \text{ \AA}$
☐ (d) — Red.

16. The wavelength of the visible He-Ne laser line is

- (a) $6328 \text{ \AA}$ (b) $5460 \text{ \AA}$ (c) $1.15 \mu\text{m}$ (d) $6943 \text{ \AA}$
☐ (a) — Red.

17. The ruby laser is a

- (a) four-level laser (b) five-level laser (c) two-level laser (d) three-level laser
☐ (d) — Lower level is ground state.

18. The He-Ne laser is a

- (a) three-level laser (b) four-level laser (c) one-level laser (d) two-level laser
☐ (b) — Lower laser level above ground.

19. The ruby laser output is

- (a) continuous (b) pulsed (c) both (d) none
☐ (b) — Three-level, high threshold.

20. The He-Ne laser output is

- (a) continuous (b) incoherent (c) pulsed (d) only infrared
☐ (a) — Four-level, CW.

21. In the He-Ne laser, the lasing atoms are

- (a) helium (b) both (c) argon (d) neon
☐ (d) — Ne transitions.

22. The role of helium in the He-Ne laser is

- (a) to produce laser light (b) to excite neon atoms by resonant collisions (c) to cool the tube (d) to absorb light
☐ (b) — Energy transfer.



- 23.** The ratio of He to Ne in a He-Ne laser is about
(a) 10 : 1 (b) 1 : 1 (c) 100 : 1 (d) 1 : 10
☐ (a) — Helium in excess.
- 24.** The optical resonator of a laser consists of
(a) a prism (b) a grating (c) two lenses (d) two mirrors, one fully and one partially reflecting
☐ (d) — Feedback.
- 25.** A four-level laser is better than a three-level laser because
(a) it is pulsed (b) the lower laser level is nearly empty (c) it has fewer levels (d) it needs more pumping
☐ (b) — Low threshold.
- 26.** In a three-level laser, population inversion requires exciting
(a) more than half the atoms (b) all atoms (c) no atoms (d) a few atoms
☐ (a) — Ground state is lower level.
- 27.** Laser light is highly monochromatic because
(a) the medium is transparent (b) stimulated emission produces identical photons and the cavity selects frequencies (c) the mirrors are coloured (d) it is intense
☐ (b) — Narrow line.
- 28.** Coherence length is
(a) $\lambda\tau_c$ (b) τ_c/c (c) c/τ_c (d) $c\tau_c$
☐ (d) — Distance over which phase is maintained.
- 29.** The coherence length of white light is
(a) metres (b) kilometres (c) micrometres (d) infinite
☐ (c) — Very broad spectrum.
- 30.** The divergence of a laser beam is approximately
(a) λ/d (b) λ^2/d (c) λd (d) d/λ
☐ (a) — Diffraction-limited.
- 31.** At room temperature, for visible light the rate of spontaneous emission compared with stimulated emission is
(a) much larger (b) much smaller (c) equal (d) zero
☐ (a) — $e^{h\nu/kT} - 1$ is huge.
- 32.** Laser action requires
(a) thermal equilibrium (b) no mirrors (c) $N_1 > N_2$ (d) population inversion
☐ (d) — Essential condition.
- 33.** The excited helium levels in the He-Ne laser are
(a) ionised (b) metastable (c) ground states (d) short-lived
☐ (b) — Transitions to ground forbidden.
- 34.** Which laser is used in optical fibre communication?
(a) He-Ne (b) CO_2 (c) Semiconductor diode laser (d) Ruby
☐ (c) — 1.3 and 1.55 μm .
- 35.** Which laser is widely used for cutting and welding?
(a) CO_2 (b) Ruby (c) Dye (d) He-Ne
☐ (a) — High-power IR.
- 36.** The efficiency of a ruby laser is about
(a) 0.1% (b) 50% (c) 90% (d) 10%
☐ (a) — Very low.



- 37.** The energy of a He-Ne laser photon is about
(a) 3.1 eV (b) 12.4 eV (c) 1.96 eV (d) 0.8 eV
□ (c) — $12400/6328$.
- 38.** Brewster windows in a He-Ne laser give output that is
(a) plane polarised (b) incoherent (c) unpolarised (d) circularly polarised
□ (a) — Polarisation selection.
- 39.** Holography uses the property of laser light called
(a) divergence (b) coherence (c) intensity (d) pulsing
□ (b) — Interference recording.
- 40.** The number of photons emitted per second by a laser of power P and wavelength λ is
(a) Phc/λ (b) $P\lambda/hc$ (c) $hc/P\lambda$ (d) $P/h\lambda$
□ (b) — $P/(hc/\lambda)$.



Short-Answer Questions (2 marks each) with Answers

Module I: Oscillations

Q1. Define simple harmonic motion.

Periodic motion in which the acceleration is proportional to the displacement from the mean position and always directed towards it: $a = -\omega^2 x$.

Q2. Write the differential equation of a damped harmonic oscillator and explain each term.

$m \ddot{x} + b \dot{x} + kx = 0$. $m\ddot{x}$ is the inertial term, $b\dot{x}$ the damping (resistive) force proportional to velocity, kx the restoring force.

Q3. Distinguish between heavy, critical and light damping.

Heavy ($\gamma > \omega_0$): slow non-oscillatory return. Critical ($\gamma = \omega_0$): fastest non-oscillatory return. Light ($\gamma < \omega_0$): oscillations with exponentially decaying amplitude.

Q4. Define relaxation time.

The time in which the energy of a damped oscillator decays to $1/e$ of its initial value: $\tau = 1/(2\gamma) = m/b$.

Q5. Define quality factor. What does a high Q mean?

$Q = 2\pi \times (\text{energy stored})/(\text{energy lost per cycle}) = \omega\tau$. A high Q means low damping, slow energy loss and a sharp resonance.

Q6. What is logarithmic decrement?

The natural log of the ratio of two successive amplitudes separated by one period: $\lambda = \ln(A_n/A_{n+1}) = \gamma T$.

Q7. What is resonance?

The phenomenon in which the amplitude of a forced oscillator becomes maximum when the driving frequency is close to its natural frequency.

Q8. Give the mechanical analogues of L, C and R.

$L \leftrightarrow \text{mass } m$, $1/C \leftrightarrow \text{spring constant } k$, $R \leftrightarrow \text{damping coefficient } b$.

Q9. What is sharpness of resonance? How does it depend on damping?

The rate at which the amplitude falls as the driving frequency moves away from resonance; measured by $Q = \omega_0/\Delta\omega$. Less damping gives sharper resonance.

Q10. What is the phase difference between the driving force and displacement at resonance?

$\pi/2$ (the displacement lags the force by 90° ; the velocity is in phase with the force).

Q11. Write the frequency of an LC circuit.



$$\nu = 1/(2\pi\sqrt{LC}).$$

Q12. What are transient and steady states in forced oscillations?

Transient: the natural damped oscillation that dies out as $e^{-\gamma t}$. Steady state: oscillation at the driving frequency with constant amplitude.

Module II: Waves and Optics

Q1. What are coherent sources? How are they obtained?

Sources with the same frequency and constant phase difference. They are obtained from a single source by division of wavefront or division of amplitude.

Q2. State the principle of superposition.

When two or more waves overlap, the resultant displacement at any point is the algebraic sum of the individual displacements.

Q3. Write the conditions for bright and dark fringes in reflected light from a thin film.

Bright: $2\mu t \cos r = (2n - 1)\lambda/2$. Dark: $2\mu t \cos r = n\lambda$.

Q4. Why does a very thin film appear dark in reflected light?

For $t \rightarrow 0$, the path difference is only $\lambda/2$ (due to reflection at the denser medium), which gives destructive interference.

Q5. Why are Newton's rings circular?

The air film thickness is constant along circles centred at the point of contact, and each ring is a locus of constant thickness.

Q6. Why is the centre of Newton's rings dark in reflected light?

At the contact point $t = 0$, but the extra $\lambda/2$ due to reflection at the glass surface gives destructive interference.

Q7. How can the refractive index of a liquid be found using Newton's rings?

Measure $(D_{n+p}^2 - D_n^2)$ with air and with the liquid: $\mu = (D_{n+p}^2 - D_n^2)_{\text{air}} / (D_{n+p}^2 - D_n^2)_{\text{liquid}}$.

Q8. Distinguish between Fresnel and Fraunhofer diffraction.

Fresnel: source/screen at finite distance, spherical wavefront, no lenses. Fraunhofer: source/screen at infinity, plane wavefront, lenses used.

Q9. State Huygens' principle.

Every point on a wavefront is a source of secondary wavelets; the new wavefront is the forward envelope of these wavelets.



Q10. Write the condition for minima in single-slit diffraction.

$$a \sin \theta = n\lambda, n = \pm 1, \pm 2, \dots$$

Q11. What is a grating element?

The distance $(a + b)$ between corresponding points of adjacent slits in a grating; $(a + b) = 1/N$, where N is the number of lines per unit length.

Q12. Write the grating equation and the maximum possible order.

$$(a + b) \sin \theta = n\lambda. n_{\max} = \text{integer part of } (a + b)/\lambda.$$

Module III: Electromagnetism

Q1. Define the gradient of a scalar field and give its physical meaning.

$\nabla \phi = \mathbf{i}\partial\phi/\partial x + \mathbf{j}\partial\phi/\partial y + \mathbf{k}\partial\phi/\partial z$. It points in the direction of the fastest increase of ϕ and is normal to the level surfaces.

Q2. What is the physical meaning of divergence?

The net outward flux per unit volume at a point; it measures the strength of a source (positive) or sink (negative).

Q3. What is the physical meaning of curl?

The circulation per unit area at a point; it measures the rotation of the field.

Q4. State Gauss's divergence theorem.

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{A} dV.$$

Q5. State Stokes' theorem.

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}.$$

Q6. Write Maxwell's equations in differential form (vacuum).

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0; \nabla \cdot \mathbf{B} = 0; \nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t; \nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\partial\mathbf{E}/\partial t.$$

Q7. What is displacement current?

$\epsilon_0\partial\mathbf{E}/\partial t$ (per unit area): a current-like term produced by a time-varying electric field, introduced by Maxwell to make Ampère's law consistent with charge conservation.

Q8. Show that EM waves are transverse.

For a plane wave along z , $\nabla \cdot \mathbf{E} = 0$ gives $ikE_z = 0 \Rightarrow E_z = 0$; similarly $B_z = 0$. So both fields are perpendicular to the direction of propagation.

Q9. Define skin depth.

The distance in a conductor over which the amplitude of an EM wave falls to $1/e$ of its surface value: $\delta = \sqrt{2/\mu\sigma\omega}$.



Q10. What is a solenoidal field? Give an example.

A field with zero divergence everywhere, e.g. the magnetic field $\mathbf{B}$ ($\nabla \cdot \mathbf{B} = 0$).

Q11. What is the speed of EM waves in vacuum in terms of μ_0 and ϵ_0 ?

$$c = 1/\sqrt{(\mu_0 \epsilon_0)} \approx 3 \times 10^8 \text{ m/s.}$$

Q12. What is the Poynting vector?

$\mathbf{S} = (1/\mu_0) \mathbf{E} \times \mathbf{B}$; it gives the energy flow per unit area per unit time (W/m^2) and the direction of propagation.

Module IV: Quantum Physics

Q1. State de Broglie's hypothesis.

Every moving material particle has a wave associated with it, of wavelength $\lambda = h/p = h/(mv)$.

Q2. Write the de Broglie wavelength of an electron accelerated through V volts.

$$\lambda = h/\sqrt{2meV} = 12.27/\sqrt{V} \text{ \AA.}$$

Q3. Define phase velocity and group velocity.

Phase velocity $v_p = \omega/k$ is the speed of individual wave crests. Group velocity $v_g = d\omega/dk$ is the speed of the envelope of a wave packet.

Q4. Show that the group velocity of a de Broglie wave equals the particle velocity.

$$\omega = E/\hbar = p^2/(2m\hbar), k = p/\hbar. v_g = d\omega/dk = dE/dp = p/m = v.$$

Q5. State the Heisenberg uncertainty principle.

It is impossible to know simultaneously the exact position and momentum of a particle: $\Delta x \Delta p \geq \hbar/2$.

Q6. What is the physical significance of the wave function?

ψ has no direct meaning; $|\psi|^2 dx$ is the probability of finding the particle between x and $x + dx$.

Q7. What is normalisation?

Scaling ψ so that $\int |\psi|^2 dx = 1$ over all space, since the particle must be somewhere.

Q8. What are eigenvalues and eigenfunctions?

If $\hat{A}\psi = a\psi$, ψ is an eigenfunction of the operator $\hat{A}$ and the constant a is the eigenvalue: the value always obtained when the observable is measured in that state.

Q9. Define expectation value.



The average value of an observable over many measurements: $\langle A \rangle = \int \psi^* \hat{A} \psi dx$.

Q10. Write the time-independent Schrödinger equation.

$$d^2\psi/dx^2 + (2m/\hbar^2)(E - V)\psi = 0.$$

Q11. Write the energy eigenvalues of a particle in a 1-D box.

$$E_n = n^2 h^2 / (8mL^2), n = 1, 2, 3, \dots$$

Q12. Why can the ground-state energy of a particle in a box not be zero?

$n = 0$ gives $\psi = 0$ (no particle). Also, by the uncertainty principle, confining the particle gives it a non-zero momentum spread, so a minimum (zero-point) energy $h^2/(8mL^2)$ exists.

Module V: Lasers

Q1. What is stimulated emission?

Emission of a photon by an excited atom triggered by an incident photon of the same energy; the emitted photon has the same frequency, phase, direction and polarisation.

Q2. Write the relations between Einstein's coefficients.

$$B_{12} = B_{21}; A_{21}/B_{21} = 8\pi h\nu^3/c^3.$$

Q3. What is population inversion? Why is it necessary?

A state with more atoms in the upper level than in the lower level ($N_2 > N_1$). It is necessary so that stimulated emission exceeds absorption, giving amplification.

Q4. What is a metastable state?

An excited state with a long lifetime ($\approx 10^{-3}$ s) compared with ordinary excited states ($\approx 10^{-8}$ s). It allows atoms to accumulate for population inversion.

Q5. What is pumping? Name two methods.

Supplying energy to create population inversion. Methods: optical pumping (ruby), electrical discharge (He-Ne), injection current (diode laser).

Q6. Mention four characteristics of laser light.

Monochromatic, coherent, highly directional and highly intense.

Q7. What is the role of the optical resonator?

It provides feedback by reflecting photons back and forth through the active medium, building up stimulated emission along the axis and producing a directional coherent beam.

Q8. Why is a four-level laser more efficient than a three-level laser?



The lower laser level is above the ground state and nearly empty, so population inversion is achieved with far less pumping.

Q9. What is the role of helium in the He-Ne laser?

Helium atoms are excited by the discharge to metastable levels and transfer their energy to neon atoms by resonant collisions, pumping the upper laser levels of neon.

Q10. Write the wavelengths of ruby and He-Ne lasers.

Ruby: 6943 Å. He-Ne: 6328 Å (visible), plus IR lines at 1.15 μm and 3.39 μm.

Q11. Give four applications of lasers.

Optical fibre communication, cutting and welding, eye surgery, barcode scanning (also holography, range finding, CD/DVD).

Long-Answer Question Bank with Answer Outlines

Use these outlines to plan 6-10 mark answers. Each outline lists what the examiner expects to see; the full content is in the section named.

Module I

Q1. What is a damped harmonic oscillator? Derive its equation of motion and discuss heavy, critical and light damping. (Sections 1.4, 1.14 B)

Define the damping force $-bv$; write $m\ddot{x} + b\dot{x} + kx = 0$; trial solution $e^{\alpha t}$; three cases with solutions; graphs (Figure 1.1); examples: door closer, galvanometer, pendulum in air.

Q2. Obtain an expression for energy decay in a damped oscillator and define relaxation time and quality factor. (Sections 1.5, 1.6, 1.14 C)

$E = (1/2)kA_0^2 e^{-2\gamma t}$; $\tau = 1/2\gamma$; $Q = 2\pi E/\Delta E = \omega\tau$; physical meaning of high and low Q ; numerical example.

Q3. Discuss forced vibrations of a damped oscillator. Obtain the amplitude and phase in the steady state and explain resonance. (Sections 1.7-1.10, 1.14 D, E)

Equation with $F_0 \cos \omega t$; transient + steady state; $A(\omega)$ and $\tan \delta$; behaviour at low, resonant and high frequencies (table); $\omega_r = \sqrt{(\omega_0^2 - 2\gamma^2)}$; resonance curves (Figure 1.2); sharpness and Q .

Q4. Explain the mechanical-electrical analogy using the series LCR circuit and discuss electrical resonance. (Sections 1.3, 1.11, 1.14 G)

KVL equation; analogy table; natural frequency; damped discharge condition R vs $2\sqrt{L/C}$; forced LCR, impedance, resonance, Q and voltage magnification.

Module II

Q1. What are coherent sources? Explain the two methods of producing them with examples. (Sections 2.4, 2.15)

Definition; why independent sources are incoherent; division of wavefront (Young, biprism) vs division of amplitude (thin film, Newton's rings); conditions for sustained interference.

Q2. Discuss interference in thin parallel films in reflected light and obtain the conditions for maxima and minima. (Sections 2.5, 2.13 A)

Ray diagram (Figure 2.1); path difference $2\mu t \cos r$; $\lambda/2$ phase change; bright/dark conditions; transmitted light complementary; colours of thin films; anti-reflection coatings.

Q3. Describe Newton's rings experiment to determine the wavelength of monochromatic light. How is the refractive index of a liquid found? (Sections 2.7, 2.13 B, 2.14)



Apparatus; formation of rings; $t = r^2/2R$; $D_n^2 = 4n\lambda R$; $\lambda = (D_{n+p}^2 - D_n^2)/4pR$; graph of D^2 vs n ; liquid method; why the centre is dark.

Q4. Discuss Fraunhofer diffraction at a single slit and obtain the intensity distribution. (Sections 2.10, 2.13 C)

Set-up with lenses; path difference $a \sin \theta$; phasor/integration giving $I = I_0(\sin \beta/\beta)^2$; minima, secondary maxima and their intensities; width of central maximum; Figure 2.3.

Q5. What is a diffraction grating? Obtain the grating equation and discuss the maximum number of orders and absent spectra. (Sections 2.11, 2.13 D)

Construction; grating element; $(a + b) \sin \theta = n\lambda$; intensity formula; $n_{\max}$; absent spectra condition; dispersive and resolving power; spectrometer method.

Module III

Q1. Explain gradient, divergence and curl with their physical significance. State Gauss's and Stokes' theorems. (Sections 3.2-3.6, 3.15)

Definitions in Cartesian form; scalar/vector nature; flux per volume and circulation per area; solenoidal and irrotational fields; identities; the two theorems.

Q2. Derive Maxwell's equations in differential and integral form and explain their physical meaning. (Sections 3.7-3.9)

Four laws in integral form; divergence and Stokes' theorems; displacement current and continuity; summary table; meaning of each equation.

Q3. Derive the electromagnetic wave equation in free space and show that EM waves are transverse. (Sections 3.10, 3.11, 3.18)

Maxwell's equations in vacuum; curl of curl identity; $\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \partial^2 \mathbf{E} / \partial t^2$; $c = 1/\sqrt{\mu_0 \epsilon_0}$; plane wave; $\nabla \cdot \mathbf{E} = 0 \Rightarrow E_z = 0$; $\mathbf{E} \perp \mathbf{B} \perp \mathbf{k}$; $\mathbf{E} = c\mathbf{B}$; Figure 3.1.

Q4. Discuss the propagation of EM waves in a conducting medium and define skin depth. (Sections 3.12, 3.19)

$\mathbf{J} = \sigma \mathbf{E}$; modified wave equation; complex k ; attenuation $e^{-z/\delta}$; $\delta = \sqrt{2/(\mu\sigma\omega)}$; numerical values for copper; applications (shielding, skin effect).

Module IV

Q1. Explain wave-particle duality. State de Broglie's hypothesis and obtain expressions for the de Broglie wavelength. (Sections 4.1, 4.2, 4.14, 4.15)

Evidence for particle nature (photoelectric, Compton) and wave nature (electron diffraction); $\lambda = h/p$; forms in terms of E and V ; $12.27/\sqrt{V}$ Å; Davisson-Germer verification; Bohr condition.

Q2. Distinguish between phase velocity and group velocity. Show that the group velocity of a de Broglie wave equals the particle velocity. (Section 4.3)

Definitions; superposition of two waves; wave packet (Figure 4.1); $v_g = v_p - \lambda \frac{dv_p}{d\lambda}$; for matter waves $v_g = v$ and $v_p v_g = c^2$.

Q3. What is a wave function? Give its physical interpretation and the conditions it must satisfy. Explain normalisation, operators and expectation values. (Sections 4.5-4.8)

Born interpretation; well-behaved conditions; normalisation integral; operator table; eigenvalue equation; expectation value formula with example.

Q4. Derive Schrödinger's time-dependent and time-independent equations. (Sections 4.9, 4.10)

Free-particle wave; E and p operators; $E = p^2/2m + V$; separation of variables; stationary states.

Q5. Solve the Schrödinger equation for a particle in a one-dimensional box. Obtain the energy eigenvalues and normalised eigenfunctions. (Section 4.11)

Potential; $\psi = 0$ outside; general solution; boundary conditions; $k = n\pi/L$; $E_n = n^2 h^2 / 8mL^2$; normalisation $\sqrt{2/L}$; zero-point energy; nodes; Figure 4.2.

Module V**Q1. Explain absorption, spontaneous emission and stimulated emission. Derive the relation between Einstein's coefficients. (Sections 5.1, 5.3, 5.13)**

Three processes with rates; comparison table; equilibrium condition; Boltzmann ratio; comparison with Planck's law; $B_{12} = B_{21}$; $A/B = 8\pi h\nu^3/c^3$; consequences.

Q2. What is population inversion? Why is it not possible in a two-level system? Explain three-level and four-level pumping schemes. (Sections 5.4, 5.8, 5.9)

Definition; metastable state; two-level argument; three-level scheme and drawback; four-level scheme and advantage; comparison table; Figure 5.1.

Q3. Describe the construction and working of a ruby laser with an energy-level diagram. Mention its drawbacks. (Section 5.10)

Ruby rod ($\text{Al}_2\text{O}_3:\text{Cr}^{3+}$); flash lamp; mirrors; three-level working; 6943 Å pulsed output; drawbacks; uses.

Q4. Describe the construction and working of a He-Ne laser. (Section 5.11)

Discharge tube, gas ratio, mirrors, Brewster windows; He excitation; resonant transfer; Ne laser lines 6328 Å, 1.15 μm, 3.39 μm; emptying of lower levels; CW output; advantages; Figure 5.2.

Q5. Explain the characteristics of laser light and list applications of lasers. (Sections 5.2, 5.12)



Monochromaticity, directionality, coherence (temporal and spatial, coherence length), intensity; applications in communication, industry, medicine, science and consumer devices.



Mixed MCQ Revision Test (All Modules)

Attempt these 40 questions in 30 minutes, then check the answers.

1. A particle in SHM has amplitude 4 cm. At what displacement is its speed half the maximum speed?

- (a) $\sqrt{2}$ cm (b) $2\sqrt{3}$ cm (c) 3 cm (d) 2 cm

☐ (b) — $v = \omega\sqrt{A^2 - x^2} = A\omega/2 \Rightarrow x^2 = 3A^2/4$.

2. The time period of a simple pendulum on a planet where g is 4 times that on Earth is

- (a) $2T$ (b) $T/2$ (c) $T/4$ (d) $4T$

☐ (b) — $T \propto 1/\sqrt{g}$.

3. In an LC circuit, the energy oscillates between

- (a) resistor and inductor (b) source and resistor (c) capacitor and resistor (d) capacitor and inductor

☐ (d) — Electric $\leftrightarrow$ magnetic.

4. If the damping of an oscillator is doubled, its relaxation time

- (a) halves (b) stays same (c) doubles (d) becomes four times

☐ (a) — $\tau = 1/2\gamma$.

5. An oscillator with $Q = 100$ loses what fraction of its energy per cycle?

- (a) 0.1% (b) 1% (c) 6.3% (d) 10%

☐ (c) — $\Delta E/E = 2\pi/Q$.

6. For a series LCR circuit at resonance, the power factor is

- (a) 1 (b) 0.707 (c) 0.5 (d) 0

☐ (a) — Purely resistive.

7. A forced oscillator driven far above resonance has amplitude controlled by

- (a) damping (b) mass (c) stiffness (d) gravity

☐ (b) — $A \approx F_0/m\omega^2$.

8. Two waves $y_1 = 3 \sin \omega t$ and $y_2 = 4 \cos \omega t$ superpose. The resultant amplitude is

- (a) 5 (b) 1 (c) 12 (d) 7

☐ (a) — 90° phase difference.

9. The intensity ratio of two coherent sources is 4 : 1. The ratio $I_{\max} : I_{\min}$ is

- (a) 16 : 1 (b) 5 : 3 (c) 9 : 1 (d) 4 : 1

☐ (c) — $(2 + 1)^2/(2 - 1)^2$.

10. The fringe width in an air wedge increases when

- (a) the wedge angle decreases (b) λ decreases (c) a liquid is introduced (d) the wedge angle increases

☐ (a) — $\beta = \lambda/2\theta$.

11. In Newton's rings, the 16th dark ring has diameter 0.4 cm. The 4th dark ring has diameter

- (a) 0.2 cm (b) 0.8 cm (c) 1.6 cm (d) 0.1 cm

☐ (a) — $D \propto \sqrt{n}$.

12. Newton's rings in transmitted light have a centre that is

- (a) bright (b) coloured (c) dark (d) invisible

☐ (a) — Complementary.

13. In single-slit diffraction, the first minimum is at 30° for $\lambda = 5000 \text{ \AA}$. The slit width is

- (a) $2 \mu\text{m}$ (b) $1 \mu\text{m}$ (c) $2.5 \times 10^{-7} \text{ m}$ (d) $5000 \text{ \AA}$

☐ (b) — $a = \lambda/\sin 30^\circ$.



- 14.** A grating with 5000 lines/cm is illuminated by $\lambda = 6000 \text{ \AA}$. The highest order visible is
(a) 2 (b) 1 (c) 4 (d) 3
 $\square$ (d) — $(a + b)/\lambda = 2 \times 10^{-6}/6 \times 10^{-7} = 3.33$.
- 15.** If $(a + b) = 3a$ in a grating, the missing order is
(a) 3rd (b) 4th (c) 2nd (d) none
 $\square$ (a) — $n = 3m$.
- 16.** The divergence of the unit radial vector $\mathbf{r}/r$ is
(a) $2/r$ (b) $3/r$ (c) 0 (d) $1/r$
 $\square$ (a) — $\nabla \cdot (\mathbf{r}/r) = 3/r - 1/r$.
- 17.** Which of these is irrotational?
(a) $\mathbf{A} = x\mathbf{i} + y\mathbf{j}$ (b) $\mathbf{A} = y\mathbf{i}$ (c) $\mathbf{A} = z\mathbf{j} - y\mathbf{k}$ (d) $\mathbf{A} = -y\mathbf{i} + x\mathbf{j}$
 $\square$ (a) — It is $\nabla((x^2 + y^2)/2)$.
- 18.** The integral form of $\nabla \cdot \mathbf{B} = 0$ is
(a) $\oint \mathbf{B} \cdot d\mathbf{l} = 0$ (b) $\iiint \mathbf{B} dV = 0$ (c) $\mathbf{B} = 0$ (d) $\oint \mathbf{B} \cdot d\mathbf{S} = 0$
 $\square$ (d) — Divergence theorem.
- 19.** Displacement current density in vacuum is
(a) $\epsilon_0 \partial E / \partial t$ (b) $\mu_0 \partial H / \partial t$ (c) ρv (d) σE
 $\square$ (a) — Maxwell's term.
- 20.** The wave equation for E in a conducting medium contains an extra first-derivative term because of
(a) magnetic monopoles (b) displacement current (c) conduction current (d) zero permittivity
 $\square$ (c) — $\mu \sigma \partial E / \partial t$.
- 21.** The skin depth of copper at 60 Hz is about
(a) 8.5 cm (b) 85 m (c) 8.5 μm (d) 8.5 mm
 $\square$ (d) — $\delta = \sqrt{2/\mu \sigma \omega}$.
- 22.** If the frequency is increased four times, the skin depth becomes
(a) 1/4 (b) 4 times (c) 1/2 (d) 2 times
 $\square$ (c) — $\delta \propto 1/\sqrt{\omega}$.
- 23.** The ratio of E and H amplitudes in free space is about
(a) 377Ω (b) 1Ω (c) 120Ω (d) $3 \times 10^8 \Omega$
 $\square$ (a) — $\sqrt{(\mu_0/\epsilon_0)}$.
- 24.** An electron and a photon have the same wavelength. Which has greater momentum?
(a) Electron (b) Photon (c) Equal (d) Depends on energy
 $\square$ (c) — $p = h/\lambda$ for both.
- 25.** The de Broglie wavelength of an electron accelerated through 10 kV is about
(a) $1.23 \text{ \AA}$ (b) $12.3 \text{ \AA}$ (c) $0.123 \text{ \AA}$ (d) $0.0123 \text{ \AA}$
 $\square$ (c) — $12.27/\sqrt{10000}$.
- 26.** The momentum of a photon of energy 3 eV is about
(a) $4.8 \times 10^{-19} \text{ kg m/s}$ (b) $9.1 \times 10^{-31} \text{ kg m/s}$ (c) $3 \times 10^8 \text{ kg m/s}$ (d) $1.6 \times 10^{-27} \text{ kg m/s}$
 $\square$ (d) — $p = E/c = 4.8 \times 10^{-19}/3 \times 10^8$.
- 27.** The uncertainty in momentum of an electron confined to 1 nm is at least about
(a) $5 \times 10^{-26} \text{ kg m/s}$ (b) $5 \times 10^{-16} \text{ kg m/s}$ (c) $5 \times 10^{-34} \text{ kg m/s}$ (d) $5 \times 10^{-20} \text{ kg m/s}$
 $\square$ (a) — $\hbar/(2 \times 10^{-9})$.

- 28.** For $\psi = A \sin(\pi x/L)$ in $0 < x < L$, the normalisation constant is
(a) $\sqrt{2/L}$ (b) $1/L$ (c) $\sqrt{L}$ (d) $2/L$
□ (a) — $A^2 L/2 = 1$.
- 29.** The eigenvalue of the operator d^2/dx^2 for $\psi = e^{3x}$ is
(a) 9 (b) e^3 (c) 3 (d) 6
□ (a) — $d^2(e^{3x})/dx^2 = 9e^{3x}$.
- 30.** If ψ_1 and ψ_2 are solutions of Schrödinger's equation, then $c_1\psi_1 + c_2\psi_2$ is
(a) not a solution (b) also a solution (c) a solution only if $c_1 = c_2$ (d) zero
□ (b) — Linearity.
- 31.** An electron in a box has $E_1 = 5$ eV. The energy needed to raise it from $n = 1$ to $n = 3$ is
(a) 15 eV (b) 40 eV (c) 10 eV (d) 45 eV
□ (b) — $9E_1 - E_1 = 8E_1$.
- 32.** The probability of finding a particle at the centre of a box in the $n = 2$ state is
(a) zero (b) 1 (c) maximum (d) $1/2$
□ (a) — Node at $L/2$.
- 33.** Increasing the mass of the particle in a box makes the energy levels
(a) closer together (b) farther apart (c) negative (d) unchanged
□ (a) — $E \propto 1/m$.
- 34.** The number of photons per second in a 3 mW laser of wavelength 6600 Å is about
(a) 10^8 (b) 10^{12} (c) 10^{20} (d) 10^{16}
□ (d) — $P/(hc/\lambda) = 3 \times 10^{-3}/3 \times 10^{-19}$.
- 35.** In thermal equilibrium at room temperature, for optical frequencies, N_2/N_1 is
(a) close to 1 (b) greater than 1 (c) extremely small (d) exactly $1/2$
□ (c) — e^{-80} .
- 36.** A laser cavity 1.5 m long has longitudinal mode spacing
(a) 50 MHz (b) 200 MHz (c) 100 MHz (d) 1 GHz
□ (c) — $c/2L$.
- 37.** In a ruby laser, the non-radiative transition is from
(a) E_1 to E_3 (b) E_2 to E_3 (c) E_2 to E_1 (d) E_3 to E_2
□ (d) — Pump band to metastable level.
- 38.** The He-Ne laser line at 3.39 μm arises from
(a) $5s \rightarrow 4p$ (b) $5s \rightarrow 3p$ (c) $2s \rightarrow 1s$ (d) $3s \rightarrow 3p$
□ (a) — IR transition.
- 39.** Temporal coherence is related to
(a) polarisation (b) beam divergence (c) spectral line width (d) intensity
□ (c) — $\tau_c \approx 1/\Delta\nu$.
- 40.** The main advantage of a semiconductor laser is
(a) visible output only (b) very high power (c) long coherence length (d) compact size and direct electrical pumping
□ (d) — Injection pumping.



Model Question Paper with Solutions

Practice pattern · Time: 3 hours · Part A: 10 short questions (2 marks each) · Part B: long questions. Follow the exact pattern announced by your university.

Note: This is a practice paper written to match the syllabus. It is not an official university question paper.

Part A

Q1. What is the Q-factor of an oscillator with $\omega_0 = 1000$ rad/s and $\gamma = 5 \text{ s}^{-1}$?

$$Q = \omega_0 / (2\gamma) = 100.$$

Q2. Find the frequency of an LC circuit with $L = 1$ mH and $C = 1$ nF.

$$\omega = 1/\sqrt{LC} = 10^6 \text{ rad/s}; \nu \approx 159 \text{ kHz}.$$

Q3. Why is a thin film dark when its thickness approaches zero (in reflected light)?

The only path difference is $\lambda/2$ due to the phase change on reflection, so the two rays cancel.

Q4. A grating has 5000 lines/cm. Find the maximum order for $\lambda = 6000 \text{ \AA}$.

$$(a + b)/\lambda = 2 \times 10^{-6} / 6 \times 10^{-7} = 3.33 \Rightarrow 3.$$

Q5. Find $\nabla \cdot \mathbf{r}$.

$$3.$$

Q6. What is the significance of $\nabla \cdot \mathbf{B} = 0$?

Magnetic monopoles do not exist; magnetic field lines are closed.

Q7. Find the de Broglie wavelength of an electron accelerated through 150 V.

$$12.27/\sqrt{150} = 1.0 \text{ \AA}.$$

Q8. Write the normalised wave function of a particle in a box.

$$\psi_n = \sqrt{2/L} \sin(n\pi x/L).$$

Q9. Why can a two-level system not produce laser action?

Since $B_{12} = B_{21}$, optical pumping can at best equalise populations; inversion ($N_2 > N_1$) cannot be achieved.

Q10. What is the wavelength of the He-Ne laser (visible line)?

$$6328 \text{ \AA}.$$

Part B

Problem 1. Derive the equation of motion of a damped harmonic oscillator and discuss the case of light damping. A damped oscillator has $m = 0.1$ kg, $k = 10$ N/m, $b = 0.02$ kg/s. Find ω' and Q .

Equation: $m\ddot{x} + b\dot{x} + kx = 0 \Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$. For $\gamma < \omega_0$: $x = A_0 e^{-\gamma t} \cos(\omega' t + \phi)$, $\omega' = \sqrt{(\omega_0^2 - \gamma^2)}$ (see Sections 1.4 and 1.14).

$\omega_0 = \sqrt{10/0.1} = 10$ rad/s; $\gamma = b/2m = 0.1$ s⁻¹; $\omega' = \sqrt{100 - 0.01} \approx 9.9995$ rad/s.

$Q = \omega_0/2\gamma = 10/0.2 = 50$.

Answer: $\omega' \approx 10$ rad/s; $Q = 50$.

Problem 2. Obtain the steady-state amplitude of a forced damped oscillator and the condition for amplitude resonance.

From Section 1.8: $A = f_0/\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$, $\tan \delta = 2\gamma\omega/(\omega_0^2 - \omega^2)$.

$dA/d\omega = 0 \Rightarrow \omega_r = \sqrt{(\omega_0^2 - 2\gamma^2)}$; $A_{\max} = f_0/(2\gamma\sqrt{(\omega_0^2 - \gamma^2)})$.

Answer: As derived; resonance slightly below ω_0 .

Problem 3. Describe Newton's rings and show that the diameters of dark rings are proportional to the square roots of natural numbers. In a Newton's rings experiment, $D_5 = 0.336$ cm, $D_{15} = 0.590$ cm, $R = 100$ cm. Find λ .

Theory: $t = r^2/2R$; dark ring $2t = n\lambda \Rightarrow D_n^2 = 4n\lambda R \Rightarrow D_n^2 \propto \sqrt{n}$.

$\lambda = (D_{15}^2 - D_5^2)/(4 \times 10 \times 1) = (3.481 - 1.129) \times 10^{-5}/40 = 5.88 \times 10^{-7}$ m.

Answer: $\lambda \approx 5880$ Å.

Problem 4. Discuss Fraunhofer diffraction at a single slit and find the positions of minima. For $a = 0.1$ mm, $\lambda = 6000$ Å, $f = 1$ m, find the width of the central maximum.

$I = I_0(\sin \beta/\beta)^2$, minima at $a \sin \theta = n\lambda$ (Section 2.10).

Width = $2\lambda f/a = 2 \times 6 \times 10^{-7} \times 1/10^{-4} = 0.012$ m.

Answer: 1.2 cm.

Problem 5. State and prove the differential form of Maxwell's equations from the integral laws. Derive the EM wave equation in vacuum and show $c = 1/\sqrt{(\mu_0 \epsilon_0)}$.

Apply the divergence theorem to Gauss's laws and Stokes' theorem to Faraday's and Ampère-Maxwell laws (Section 3.9).

Curl of Faraday's law with $\nabla \cdot \mathbf{E} = 0$ gives $\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \partial^2 \mathbf{E} / \partial t^2$, so $v = 1/\sqrt{(\mu_0 \epsilon_0)} = 3 \times 10^8$ m/s (Section 3.10).

Answer: As derived.

Problem 6. Find the divergence and curl of $\mathbf{A} = x^2 y \mathbf{i} + y^2 z \mathbf{j} + z^2 x \mathbf{k}$ at $(1, 1, 1)$.

$\nabla \cdot \mathbf{A} = 2xy + 2yz + 2zx = 6$ at $(1, 1, 1)$.

$\nabla \times \mathbf{A} = (0 - y^2) \mathbf{i} + (0 - z^2) \mathbf{j} + (0 - x^2) \mathbf{k} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ at $(1, 1, 1)$.

Answer: $\nabla \cdot \mathbf{A} = 6$; $\nabla \times \mathbf{A} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Problem 7. Solve the Schrödinger equation for a particle in a one-dimensional infinite well. Find the first two energies of an electron in a box of width 5 Å.

$$E_n = n^2 h^2 / (8mL^2), \psi_n = \sqrt{2/L} \sin(n\pi x/L) \text{ (Section 4.11).}$$

$$E_1 = 37.6 \text{ eV} \times (1/5)^2 = 1.50 \text{ eV}; E_2 = 6.02 \text{ eV}.$$

Answer: $E_1 \approx 1.5 \text{ eV}$; $E_2 \approx 6.0 \text{ eV}$.

Problem 8. Distinguish between phase velocity and group velocity and show that $v_g = v_p - \lambda dv_p/d\lambda$.

$$\omega = v_p k \Rightarrow v_g = d\omega/dk = v_p + k dv_p/dk. \text{ With } k = 2\pi/\lambda, dk = -(2\pi/\lambda^2)d\lambda, \text{ so } k dv_p/dk = -\lambda dv_p/d\lambda.$$

Answer: $v_g = v_p - \lambda dv_p/d\lambda$.

Problem 9. Derive the relation between Einstein's coefficients. Find the ratio of spontaneous to stimulated emission for $\lambda = 5000 \text{ Å}$ at 1000 K .

Derivation: Section 5.13.

$$h\nu/k_B T = (12400/5000 \text{ eV}) / (8.617 \times 10^{-5} \times 1000 \text{ eV}) = 2.48/0.08617 = 28.8. \text{ Ratio} = e^{28.8} - 1 \approx 3.2 \times 10^{12}.$$

Answer: $\approx 3 \times 10^{12}$.

Problem 10. With a neat energy-level diagram, explain the construction and working of the He-Ne laser.

See Section 5.11 and Figure 5.2: He:Ne $\approx 10:1$ discharge tube; He excited to 19.81/20.61 eV metastable levels; resonant transfer to Ne 3s/5s levels; lasing 5s $\rightarrow$ 3p at 6328 Å; lower levels emptied by spontaneous emission and wall collisions; continuous output.

Answer: As described.



Formula Sheet (Last-Minute Revision)

Oscillations

SHM: $\ddot{x} + \omega^2 x = 0$; $x = A \sin(\omega t + \phi)$; $v = \omega \sqrt{A^2 - x^2}$; $E = (1/2)m\omega^2 A^2$
 Spring $T = 2\pi\sqrt{m/k}$; pendulum $T = 2\pi\sqrt{l/g}$; LC $\omega_0 = 1/\sqrt{LC}$
 Damped: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$, $\gamma = b/2m$ (mech), $R/2L$ (elec); $\omega' = \sqrt{\omega_0^2 - \gamma^2}$
 Heavy $\gamma > \omega_0$; critical $\gamma = \omega_0$ ($R = 2\sqrt{L/C}$); light $\gamma < \omega_0$
 $E = E_0 e^{-2\gamma t}$; $\tau = 1/2\gamma$; $Q = \omega\tau = \omega_0/2\gamma = \omega_0 L/R$; log decrement $\lambda = \gamma T$
 Forced: $A = f_0/\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$; $\tan \delta = 2\gamma\omega/(\omega_0^2 - \omega^2)$; $\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$
 $P_{\max} = F_0^2/2b$ at ω_0 ; $\Delta\omega = 2\gamma$; $Q = \omega_0/\Delta\omega$; LCR resonance $Z = R$, $V_L = QV$

Waves and optics

$v = \nu\lambda = \omega/k$; $y = A \sin(\omega t - kx)$; string $v = \sqrt{T/\mu}$; $\partial^2 y/\partial t^2 = v^2 \partial^2 y/\partial x^2$
 $R^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos \delta$; $I_{\max}/I_{\min} = ((a_1 + a_2)/(a_1 - a_2))^2$
 Thin film (reflected): dark $2\mu t \cos r = n\lambda$; bright $2\mu t \cos r = (2n - 1)\lambda/2$
 Wedge $\beta = \lambda/2\mu\theta$; AR coating $t = \lambda/4\mu$
 Newton's rings: $D_n^2 = 4n\lambda R$ (dark); $\lambda = (D_{n+p}^2 - D_n^2)/4pR$; $\mu = (\Delta D^2)_{\text{air}}/(\Delta D^2)_{\text{liq}}$
 Single slit: $I = I_0(\sin \beta/\beta)^2$, $\beta = \pi a \sin \theta/\lambda$; minima $a \sin \theta = n\lambda$; central width $2\lambda f/a$
 Grating: $(a + b) \sin \theta = n\lambda$; $(a + b) = 1/N$; $n_{\max} = (a + b)/\lambda$; $R = nN$

Electromagnetism

$\nabla\phi$ (vector); $\nabla \cdot \mathbf{A}$ (scalar); $\nabla \times \mathbf{A}$ (vector); $\nabla \times \nabla\phi = 0$; $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
 Gauss: $\oint \mathbf{A} \cdot d\mathbf{S} = \iiint \nabla \cdot \mathbf{A} dV$; Stokes: $\oint \mathbf{A} \cdot d\mathbf{l} = \iint (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$
 $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$; $\nabla \cdot \mathbf{B} = 0$; $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$; $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E}/\partial t$
 $\mathbf{J}_d = \epsilon_0 \partial \mathbf{E}/\partial t$; $\nabla \cdot \mathbf{J} + \partial \rho/\partial t = 0$
 $\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \partial^2 \mathbf{E}/\partial t^2$; $c = 1/\sqrt{\mu_0 \epsilon_0}$; $n = \sqrt{\mu_r \epsilon_r}$; $E = cB$
 $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$; $I = (1/2)c\epsilon_0 E_0^2$; skin depth $\delta = \sqrt{2/\mu\sigma\omega}$

Quantum physics

$E = h\nu = hc/\lambda$; $E(\text{eV}) = 12400/\lambda(\text{\AA})$; $p = h/\lambda$
 $\lambda = h/mv = h/\sqrt{2mE} = h/\sqrt{2meV} = 12.27/\sqrt{V} \text{\AA}$ (electron)
 $v_p = \omega/k$; $v_g = d\omega/dk = v_p - \lambda dv_p/d\lambda$; matter wave: $v_g = v$, $v_p v_g = c^2$
 $\Delta x \Delta p \geq \hbar/2$; $\Delta E \Delta t \geq \hbar/2$
 $\int |\psi|^2 dx = 1$; $\hat{p} = -i\hbar \partial/\partial x$; $\hat{E} = i\hbar \partial/\partial t$; $\langle A \rangle = \int \psi^* \hat{A} \psi dx$
 $i\hbar \partial\psi/\partial t = -(\hbar^2/2m)\partial^2\psi/\partial x^2 + V\psi$; $d^2\psi/dx^2 + (2m/\hbar^2)(E - V)\psi = 0$
 Box: $E_n = n^2\hbar^2/8mL^2$; $\psi_n = \sqrt{2/L} \sin(n\pi x/L)$; electron in 1 \AA box: $E_1 = 37.6 \text{ eV}$

Lasers

Absorption $B_{12}\rho N_1$; spontaneous $A_{21}N_2$; stimulated $B_{21}\rho N_2$
 $B_{12} = B_{21}$; $A_{21}/B_{21} = 8\pi h\nu^3/c^3$; spontaneous/stimulated $= e^{h\nu/kT} - 1$



$$N_2/N_1 = e^{-(E_2 - E_1)/k_B T}; \text{ inversion } N_2 > N_1$$

Coherence length $L_c = c\tau_c = \lambda^2/\Delta\lambda$; divergence $\theta \approx \lambda/d$; mode spacing $c/2L$

Ruby 6943 Å (3-level, pulsed); He-Ne 6328 Å (4-level, CW); photons/s = $P\lambda/hc$