



GANDHI SCHOOL OF ENGINEERING

Brahmapur, Odisha

E-Notes

Mathematics-I

Complete E-Notes

B.Tech 1st Year · Semester 1 · Circuit Branch

Academic Year 2026-27



How to use these notes

These notes follow the official 2026-27 syllabus for Mathematics-I (3-0-0, 3 credits, 40 contact hours, 100 university marks + 50 internal). Each module has five parts:

- **Theory:** definitions, theorems and the reasoning behind them, in exam language.
- **Formula box:** everything you must memorise, collected in shaded boxes.
- **Solved problems:** worked step by step in the style expected in the university exam.
- **Practice problems:** with final answers, so you can check yourself.
- **MCQs:** each with the correct option and a one-line reason.

A formula sheet for last-minute revision is at the end.

Module	Topic	Hours
I	Basic Calculus: length, surface area, volume of revolution, improper integrals, Beta and Gamma functions	8
II	Single-variable Calculus: Rolle's theorem, Mean Value Theorems, extreme values, Taylor and Maclaurin series	8
III	Multivariable Calculus: partial derivatives, Jacobians, Hessian, maxima/minima/saddle points, Lagrange multipliers	8
IV	Linear Algebra: vector spaces, basis and dimension, linear systems, Gauss elimination, linear dependence, rank	8
V	Linear Algebra: inverse by Gauss-Jordan, symmetric/skew/orthogonal matrices, eigenvalues and eigenvectors, Cayley-Hamilton theorem	8

Reference books (from syllabus)

- G.B. Thomas and R.L. Finney, Calculus and Analytic Geometry, Pearson.
- Erwin Kreyszig, Advanced Engineering Mathematics, Wiley.
- B.V. Ramana, Higher Engineering Mathematics, Tata McGraw Hill.
- Gilbert Strang, Introduction to Linear Algebra.
- T. Veerarajan, Engineering Mathematics for First Year, Tata McGraw Hill.



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Mathematics-I

Subject: Mathematics-I · Semester 1 · 3-0-0 · 3 credits · 40 hours · University 100 + Internal 50

Module I: Basic Calculus (Applications of Definite Integrals)

The definite integral $\int_a^b f(x) dx$ is the limit of a sum of very thin strips. Any quantity that can be broken into small pieces, added up, and made exact by a limit can be written as a definite integral. Length of a curve, area of a surface and volume of a solid are three such quantities. In each case we find the contribution of a tiny element (ds , dS or dV) and integrate it over the given range.

1.1 Review: the definite integral as a limit of sums

Divide $[a, b]$ into n equal parts of width $\Delta x = (b - a)/n$ and pick a point x_i in each part. Then

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

If $F'(x) = f(x)$, the Fundamental Theorem of Calculus gives $\int_a^b f(x) dx = F(b) - F(a)$.

Useful standard results you will use repeatedly in this module:

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = [(n-1)(n-3)\dots]/[n(n-2)\dots] \times k, \text{ where } k = \pi/2 \text{ if } n \text{ is even, } k = 1 \text{ if } n \text{ is odd (Wallis' formula)}$$

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx ; \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f \text{ is even, } = 0 \text{ if } f \text{ is odd}$$

$$\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(2a-x) = f(x), \text{ and } = 0 \text{ if } f(2a-x) = -f(x)$$

Examples of Wallis' formula: $\int_0^{\pi/2} \sin^4 x dx = (3 \cdot 1)/(4 \cdot 2) \cdot \pi/2 = 3\pi/16$; $\int_0^{\pi/2} \cos^5 x dx = (4 \cdot 2)/(5 \cdot 3 \cdot 1) = 8/15$.

1.2 Length of a plane curve (rectification)

Take a tiny piece of the curve between (x, y) and $(x + dx, y + dy)$. By Pythagoras, its length is $ds = \sqrt{(dx)^2 + (dy)^2}$. Dividing and multiplying by dx , or by dt , or by $d\theta$, gives the three standard forms.

$$\text{Cartesian } y = f(x), a \leq x \leq b: s = \int_a^b \sqrt{1 + (dy/dx)^2} dx$$

$$\text{Cartesian } x = g(y), c \leq y \leq d: s = \int_c^d \sqrt{1 + (dx/dy)^2} dy$$

$$\text{Parametric } x = x(t), y = y(t), t_1 \leq t \leq t_2: s = \int \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$$

$$\text{Polar } r = f(\theta), \alpha \leq \theta \leq \beta: s = \int_{\alpha}^{\beta} \sqrt{r^2 + (dr/d\theta)^2} d\theta$$

$$\text{Polar } \theta = g(r): s = \int \sqrt{1 + r^2(d\theta/dr)^2} dr$$

Working rule:

- Identify which form the curve is in (y as function of x , parametric, or polar).
- Differentiate, square, add 1 (or add the other term), and simplify the expression under the root. In most exam problems it becomes a perfect square.
- Use symmetry. For a closed symmetric curve, find the length in one quadrant and multiply by 4. For a cardioid, find the length for $0 \leq \theta \leq \pi$ and double it.
- Put the correct limits. For a full loop, find where the curve meets the axes.



Note: Common curves you must know: circle $x^2 + y^2 = a^2$ (perimeter $2\pi a$); astroid $x^{2/3} + y^{2/3} = a^{2/3}$ (perimeter $6a$); cardioid $r = a(1 + \cos \theta)$ (perimeter $8a$); cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ (one arch has length $8a$); catenary $y = c \cosh(x/c)$.

1.3 Surface area of a solid of revolution

When a curve is rotated about an axis, each tiny arc ds sweeps out a thin band. The band is like the side of a very short cylinder, with radius equal to the distance of the arc from the axis. So its area is $(2\pi \times \text{radius}) \times ds$.

About the x-axis: $S = \int_a^b 2\pi y \, ds = \int_a^b 2\pi y \sqrt{1 + (dy/dx)^2} \, dx$

About the y-axis: $S = \int_a^b 2\pi x \, ds = \int_a^b 2\pi x \sqrt{1 + (dy/dx)^2} \, dx$

Parametric, about x-axis: $S = \int 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt$

Polar, about the initial line: $S = \int 2\pi r \sin \theta \cdot \sqrt{r^2 + (dr/d\theta)^2} \, d\theta$

Polar, about the line $\theta = \pi/2$: $S = \int 2\pi r \cos \theta \cdot \sqrt{r^2 + (dr/d\theta)^2} \, d\theta$

The only thing that changes between the formulas is the radius of the band (y , x , $r \sin \theta$ or $r \cos \theta$) and the expression for ds .

1.4 Volume of a solid of revolution

Disc method. Rotate the region under $y = f(x)$, $a \leq x \leq b$, about the x-axis. A thin vertical strip becomes a disc of radius y and thickness dx , so $dV = \pi y^2 \, dx$.

About x-axis: $V = \int_a^b \pi y^2 \, dx$ **About y-axis:** $V = \int_c^d \pi x^2 \, dy$

Washer method. If the region lies between two curves y_1 (outer) and y_2 (inner), each strip becomes a washer (ring).

$$V = \int_a^b \pi (y_1^2 - y_2^2) \, dx$$

Shell method. Rotate the region under $y = f(x)$ about the y-axis. A thin vertical strip at distance x becomes a cylindrical shell of radius x , height y and thickness dx .

$$V = \int_a^b 2\pi x y \, dx \text{ (region rotated about the y-axis)}$$

Parametric and polar forms.

Parametric about x-axis: $V = \int \pi y^2 (dx/dt) \, dt$

Polar region bounded by $r = f(\theta)$, rotated about initial line: $V = \int_{\alpha}^{\beta} (2\pi/3) r^3 \sin \theta \, d\theta$

Polar region rotated about $\theta = \pi/2$: $V = \int_{\alpha}^{\beta} (2\pi/3) r^3 \cos \theta \, d\theta$

Theorems of Pappus (quick check). Volume = (area of region) $\times$ (distance travelled by its centroid) = $A \times 2\pi \bar{y}$. Surface area = (length of curve) $\times 2\pi \bar{y}$, where $\bar{y}$ is the distance of the centroid from the axis. For a torus formed by rotating a circle of radius a whose centre is at distance b from the axis: $V = \pi a^2 \times 2\pi b = 2\pi^2 a^2 b$, and $S = 2\pi a \times 2\pi b = 4\pi^2 ab$.

1.5 Improper integrals

The definite integral $\int_a^b f(x) \, dx$ is called **proper** when the interval $[a, b]$ is finite and f is bounded on it. If either condition fails, the integral is **improper**.

- **Type I (infinite limit):** $\int_a^\infty f(x) dx$, $\int_{-\infty}^b f(x) dx$ or $\int_{-\infty}^\infty f(x) dx$.
- **Type II (infinite integrand):** f becomes unbounded at some point of $[a, b]$, for example $\int_0^1 dx/\sqrt{x}$ ($f \rightarrow \infty$ at $x = 0$).
- **Mixed:** both, for example $\int_0^\infty dx/(\sqrt{x}(1+x))$.

Definition. An improper integral is evaluated as a limit.

$$\begin{aligned}\int_a^\infty f(x) dx &= \lim_{t \rightarrow \infty} \int_a^t f(x) dx \\ \int_{-\infty}^\infty f(x) dx &= \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx \text{ (both must converge separately)} \\ \text{If } f \text{ is unbounded at } x = a: \int_a^b f(x) dx &= \lim_{\varepsilon \rightarrow 0^+} \int_{a+\varepsilon}^b f(x) dx\end{aligned}$$

If the limit exists and is finite, the integral **converges**; otherwise it **diverges**.

Standard p-tests (very important for MCQs):

$$\begin{aligned}\int_1^\infty dx/x^p &\text{ converges if } p > 1 \text{ (value } 1/(p-1)), \text{ diverges if } p \leq 1 \\ \int_0^1 dx/x^p &\text{ converges if } p < 1 \text{ (value } 1/(1-p)), \text{ diverges if } p \geq 1 \\ \int_0^\infty e^{-ax} dx &= 1/a \text{ (} a > 0) \int_0^\infty dx/(1+x^2) = \pi/2 \int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}\end{aligned}$$

Comparison test. If $0 \leq f(x) \leq g(x)$ for $x \geq a$: if $\int g$ converges then $\int f$ converges; if $\int f$ diverges then $\int g$ diverges.

Limit comparison test. If $f, g > 0$ and $\lim f(x)/g(x) = L$ with $0 < L < \infty$, then $\int f$ and $\int g$ both converge or both diverge.

Absolute convergence. If $\int |f|$ converges, then $\int f$ converges (absolutely). Example: $\int_1^\infty \sin x / x^2 dx$ converges absolutely since $|\sin x/x^2| \leq 1/x^2$.

1.6 Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, n > 0$$

This improper integral converges for every $n > 0$. Its most important properties:

$$\begin{aligned}\Gamma(1) &= 1 \quad \Gamma(n+1) = n \Gamma(n) \quad \Gamma(n+1) = n! \text{ for } n = 0, 1, 2, \dots \\ \Gamma(1/2) &= \sqrt{\pi} \quad \Gamma(3/2) = \sqrt{\pi}/2 \quad \Gamma(5/2) = 3\sqrt{\pi}/4 \quad \Gamma(7/2) = 15\sqrt{\pi}/8 \\ \Gamma(n) \Gamma(1-n) &= \pi / \sin(n\pi), 0 < n < 1 \text{ (reflection formula)} \\ \Gamma(n) &= \Gamma(n+1)/n \text{ is used to define } \Gamma \text{ for negative non-integers, e.g. } \Gamma(-1/2) = \\ \Gamma(1/2)/(-1/2) &= -2\sqrt{\pi}\end{aligned}$$

Proof of $\Gamma(n+1) = n\Gamma(n)$. Integrate by parts: $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx = [-x^n e^{-x}]_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx = 0 + n\Gamma(n)$.

Proof of $\Gamma(1/2) = \sqrt{\pi}$. $\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2} dx$. Put $x = t^2$, $dx = 2t dt$: $\Gamma(1/2) = 2 \int_0^\infty e^{-t^2} dt = 2 \times (\sqrt{\pi}/2) = \sqrt{\pi}$.

Useful alternative forms of Gamma:

$$\begin{aligned}\int_0^\infty e^{-kx} x^{n-1} dx &= \Gamma(n)/k^n \int_0^\infty e^{-x^2} x^{2n-1} dx = \Gamma(n)/2 \\ \int_0^1 x^m (\log(1/x))^n dx &= \Gamma(n+1)/(m+1)^{n+1}\end{aligned}$$

1.7 Beta function

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, m > 0, n > 0$$

Properties:

$$\begin{aligned}
 B(m, n) &= B(n, m) \text{ (symmetry; put } x = 1 - t) \\
 B(m, n) &= \Gamma(m) \Gamma(n) / \Gamma(m + n) \text{ (relation between Beta and Gamma)} \\
 B(m, n) &= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta \text{ (trigonometric form)} \\
 \int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta &= (1/2) B((p+1)/2, (q+1)/2) = \Gamma((p+1)/2) \Gamma((q+1)/2) / [2 \Gamma((p+q+2)/2)] \\
 B(m, n) &= \int_0^\infty x^{m-1} / (1+x)^{m+n} \, dx \text{ (second form, infinite range)} \\
 B(m, n) &= B(m+1, n) + B(m, n+1) \\
 \text{Duplication formula: } \Gamma(m) \Gamma(m+1/2) &= \sqrt{\pi} \Gamma(2m) / 2^{2m-1}
 \end{aligned}$$

Proof of the trigonometric form. Put $x = \sin^2 \theta$ in the definition, so $dx = 2 \sin \theta \cos \theta \, d\theta$ and $1 - x = \cos^2 \theta$. Then $B(m, n) = \int_0^{\pi/2} \sin^{2m-2} \theta \cos^{2n-2} \theta \cdot 2 \sin \theta \cos \theta \, d\theta = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta$.

How to recognise which function to use:

- Limits 0 to ∞ with e^{-x} (or e^{-kx} , e^{-x^2}) $\rightarrow$ Gamma.
- Limits 0 to 1 with $x^a(1-x)^b \rightarrow$ Beta directly. With $x^a(1-x^n)^b$, first substitute $x^n = t$.
- Limits 0 to $\pi/2$ with powers of sin and cos $\rightarrow$ trigonometric form of Beta.
- Limits 0 to ∞ with $x^a/(1+x)^b \rightarrow$ second form of Beta.
- Limits 0 to a with $x^m(a-x)^n$, or $\sqrt{(a^2-x^2)}$, $\rightarrow$ substitute $x = at$ or $x = a \sin \theta$, then Beta.

1.8 Curve tracing: what you need before finding lengths, areas and volumes

Most mistakes in this module come from wrong limits, not wrong integration. Before applying any formula, make a rough sketch of the curve. Check the following points in order.

- **Symmetry.** If only even powers of y occur, the curve is symmetric about the x -axis (e.g. $y^2 = 4ax$). If only even powers of x occur, it is symmetric about the y -axis. If replacing (x, y) by $(-x, -y)$ leaves the equation unchanged, it is symmetric about the origin. If interchanging x and y leaves it unchanged, it is symmetric about $y = x$ (e.g. $x^3 + y^3 = 3axy$). For polar curves: if replacing θ by $-\theta$ leaves r unchanged, the curve is symmetric about the initial line (e.g. $r = a(1 + \cos \theta)$).
- **Origin.** If the equation has no constant term, the curve passes through the origin. The tangents at the origin are found by equating the lowest-degree terms to zero.
- **Intercepts.** Put $y = 0$ to find where the curve cuts the x -axis, and $x = 0$ for the y -axis.
- **Region of existence.** Find where y^2 (or r^2) becomes negative. The curve does not exist there. For $y^2 = x(x-1)^2$, $y^2 < 0$ for $x < 0$, so the curve lies in $x \geq 0$.
- **Asymptotes.** Lines that the curve approaches at infinity (e.g. $x = 2a$ for the cissoid $y^2(2a-x) = x^3$).
- **Loops.** If the curve crosses an axis twice with a closed portion in between, the limits for the loop are those crossing points (for $y^2 = x(x-1)^2$, the loop is $0 \leq x \leq 1$).

1.9 Standard curves and their shapes

Curve	Equation	Key facts
Parabola	$y^2 = 4ax$	Symmetric about x -axis; vertex at origin; latus rectum $x = a$



Curve	Equation	Key facts
Ellipse	$x^2/a^2 + y^2/b^2 = 1$	Parametric $x = a \cos t$, $y = b \sin t$; area πab
Astroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a \cos^3 t$, $y = a \sin^3 t$; four cusps; perimeter $6a$; area $3\pi a^2/8$
Cycloid	$x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$	Traced by a point on a rolling circle; arch length $8a$; arch area $3\pi a^2$
Catenary	$y = c \cosh(x/c)$	Shape of a hanging chain; arc $s = c \sinh(x/c)$
Cardioid	$r = a(1 + \cos \theta)$	Heart-shaped; symmetric about initial line; length $8a$; area $3\pi a^2/2$
Circle (polar)	$r = 2a \cos \theta$	Circle of radius a through the pole
Lemniscate	$r^2 = a^2 \cos 2\theta$	Figure-eight; two loops; exists for $-\pi/4 \leq \theta \leq \pi/4$
Equiangular spiral	$r = ae^{b\theta}$	Tangent makes a constant angle with the radius vector

1.10 Derivation of the arc-length formula (exam derivation)

Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be two neighbouring points on the curve $y = f(x)$, and let Δs be the arc PQ . For small Δx the arc is almost equal to the chord PQ , so $(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$. Dividing by $(\Delta x)^2$ gives $(\Delta s/\Delta x)^2 \approx 1 + (\Delta y/\Delta x)^2$. Taking the limit as $\Delta x \rightarrow 0$, $(ds/dx)^2 = 1 + (dy/dx)^2$, so $ds/dx = \sqrt{1 + (dy/dx)^2}$. Integrating from $x = a$ to $x = b$ gives the length $s = \int_a^b \sqrt{1 + (dy/dx)^2} dx$.

For polar curves, $x = r \cos \theta$ and $y = r \sin \theta$ with $r = f(\theta)$. Then $dx/d\theta = r' \cos \theta - r \sin \theta$ and $dy/d\theta = r' \sin \theta + r \cos \theta$. Squaring and adding, $(dx/d\theta)^2 + (dy/d\theta)^2 = r'^2 + r^2$ (the cross terms cancel). Hence $ds/d\theta = \sqrt{r^2 + (dr/d\theta)^2}$.

1.11 Derivation of the volume and surface formulas

Volume by slicing. Cut the solid into thin slices perpendicular to the x -axis. If the cross-section at x has area $A(x)$, the slice has volume $\approx A(x) \Delta x$. Adding and taking the limit: $V = \int_a^b A(x) dx$. For a solid of revolution about the x -axis, each cross-section is a circle of radius y , so $A(x) = \pi y^2$, giving $V = \int \pi y^2 dx$.

Surface by bands. Each small arc Δs , rotating about the x -axis, sweeps out the curved surface of a frustum of a cone with slant height Δs and mean radius y . Its area is $2\pi y \Delta s$. Adding and taking the limit gives $S = \int 2\pi y ds$.

Shell method. A thin strip of width Δx at distance x from the y -axis, of height y , rotated about the y -axis becomes a thin cylindrical shell. Unrolled, it is almost a rectangular sheet of length $2\pi x$, height y and thickness Δx , so $\Delta V = 2\pi xy \Delta x$.

1.12 Theorems of Pappus-Guldinus (with proof idea)

First theorem (surface). If a plane arc of length L is rotated about an axis in its plane that does not cross it, the area of the surface generated is $S = L \times 2\pi \bar{y}$, where $\bar{y}$ is the distance of the centroid of the arc from the axis.



Second theorem (volume). If a plane region of area A is rotated about an axis in its plane that does not cross it, the volume generated is $V = A \times 2\pi\bar{y}$, where $\bar{y}$ is the distance of the centroid of the region from the axis.

Proof idea. The centroid of a region is $\bar{y} = (1/A)\iint y \, dA$. The volume swept is $\iint 2\pi y \, dA = 2\pi(A\bar{y})$. The same argument with arc length gives the surface theorem.

Example. A semicircular region of radius a (area $\pi a^2/2$) has its centroid at distance $4a/(3\pi)$ from the diameter. Rotating about the diameter gives a sphere: $V = (\pi a^2/2) \times 2\pi \times 4a/(3\pi) = (4/3)\pi a^3$. $\square$

1.13 Convergence of the Gamma and Beta integrals (why $n > 0$ is needed)

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$ is improper at both ends. Split it at $x = 1$.

- On $[0, 1]$, $e^{-x} x^{n-1} \leq x^{n-1} = 1/x^{1-n}$. This converges by the p-test when $1 - n < 1$, i.e. $n > 0$.
- On $[1, \infty)$, e^{-x} decays faster than any power, so $e^{-x} x^{n-1} \leq C/x^2$ for large x . This part converges for every n .

Therefore $\Gamma(n)$ converges exactly when $n > 0$. Similarly $B(m, n)$ converges when $m > 0$ and $n > 0$ (the singularities at $x = 0$ and $x = 1$ behave like x^{m-1} and $(1-x)^{n-1}$).

Graph of $\Gamma(x)$. For $x > 0$, Γ decreases from $+\infty$ (as $x \rightarrow 0+$) to a minimum of about 0.8856 at $x \approx 1.4616$, then increases rapidly ($\Gamma(2) = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 6$). For negative non-integers, Γ is defined by $\Gamma(x) = \Gamma(x+1)/x$, and it is undefined (infinite) at $0, -1, -2, \dots$

1.14 Proof that $B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$ (outline)

Write $\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1} \, dx$ and $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} \, dy$. Multiply to get a double integral over the first quadrant: $\Gamma(m)\Gamma(n) = 4\iint e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} \, dx \, dy$. Change to polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, $dx \, dy = r \, dr \, d\theta$ (Jacobian r , see Module III):
 $\Gamma(m)\Gamma(n) = [2\int_0^\infty e^{-r^2} r^{2(m+n)-1} \, dr] \times [2\int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta \, d\theta] = \Gamma(m+n) \times B(m, n)$.

This proof uses three ideas from this course: the alternative form of Gamma, the Jacobian of polar coordinates, and the trigonometric form of Beta.

1.15 Common mistakes to avoid in Module I

- Forgetting to multiply by the symmetry factor ($\times 2$ or $\times 4$) or multiplying twice.
- Using $\int 2\pi y \, dx$ instead of $\int 2\pi y \, ds$ for surface area.
- Taking the positive square root without checking the sign on the interval (e.g. $\sqrt{(\sin^2(\theta/2))} = |\sin(\theta/2)|$).
- Writing $\Gamma(n) = n!$ instead of $(n-1)!$.
- In Beta, exponents in the integrand are $m-1$ and $n-1$, so $\int_0^1 x^3(1-x)^2 \, dx = B(4, 3)$, not $B(3, 2)$.
- Declaring $\int_{-1}^1 dx/x^2 = -2$ by blind use of the antiderivative. It is improper at 0 and diverges.

1.16 Solved problems: Module I

Problem 1. Find the length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/3$.

$$dy/dx = (\sec x \tan x)/\sec x = \tan x.$$

$$1 + (dy/dx)^2 = 1 + \tan^2 x = \sec^2 x, \text{ so } \sqrt{1 + (dy/dx)^2} = \sec x \text{ (positive on } [0, \pi/3]).$$

$$s = \int_0^{\pi/3} \sec x \, dx = [\log(\sec x + \tan x)]_0^{\pi/3} = \log(2 + \sqrt{3}) - \log(1 + 0).$$

Answer: $s = \log(2 + \sqrt{3}) \approx 1.317$ units.

Problem 2. Find the length of the curve $y = (2/3)x^{3/2}$ from $x = 0$ to $x = 3$.

$$dy/dx = x^{1/2}, \text{ so } 1 + (dy/dx)^2 = 1 + x.$$

$$s = \int_0^3 \sqrt{1 + x} \, dx = [(2/3)(1 + x)^{3/2}]_0^3 = (2/3)(8 - 1).$$

Answer: $s = 14/3$ units.

Problem 3. Find the perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

Use the parametric form $x = a \cos^3 t$, $y = a \sin^3 t$, $0 \leq t \leq 2\pi$.

$$dx/dt = -3a \cos^2 t \sin t, \, dy/dt = 3a \sin^2 t \cos t.$$

$$(dx/dt)^2 + (dy/dt)^2 = 9a^2 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t) = 9a^2 \sin^2 t \cos^2 t, \text{ so } ds/dt = 3a \sin t \cos t \text{ in the first quadrant.}$$

$$\text{The curve is symmetric in all four quadrants: } s = 4 \int_0^{\pi/2} 3a \sin t \cos t \, dt = 12a \times [\sin^2 t/2]_0^{\pi/2} = 12a \times 1/2.$$

Answer: Perimeter = $6a$.

Problem 4. Find the length of one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

One arch corresponds to $0 \leq \theta \leq 2\pi$.

$$dx/d\theta = a(1 - \cos \theta), \, dy/d\theta = a \sin \theta.$$

$$(dx/d\theta)^2 + (dy/d\theta)^2 = a^2[(1 - \cos \theta)^2 + \sin^2 \theta] = a^2(2 - 2\cos \theta) = 4a^2 \sin^2(\theta/2).$$

$$s = \int_0^{2\pi} 2a \sin(\theta/2) \, d\theta = 2a[-2 \cos(\theta/2)]_0^{2\pi} = 2a(2 + 2) = 8a.$$

Answer: Length of one arch = $8a$.

Problem 5. Find the entire length of the cardioid $r = a(1 + \cos \theta)$. Also show that the upper half is bisected by $\theta = \pi/3$.

$$dr/d\theta = -a \sin \theta. \, r^2 + (dr/d\theta)^2 = a^2[(1 + \cos \theta)^2 + \sin^2 \theta] = 2a^2(1 + \cos \theta) = 4a^2 \cos^2(\theta/2).$$

$$\text{So } ds/d\theta = 2a \cos(\theta/2) \text{ for } 0 \leq \theta \leq \pi. \text{ By symmetry about the initial line, total length } s = 2 \int_0^{\pi} 2a \cos(\theta/2) \, d\theta = 4a[2 \sin(\theta/2)]_0^{\pi} = 8a.$$

$$\text{Length of the upper half} = 4a. \text{ Length from } \theta = 0 \text{ to } \theta = \pi/3 = \int_0^{\pi/3} 2a \cos(\theta/2) \, d\theta = 4a \sin(\pi/6) = 2a, \text{ which is half of } 4a.$$

Answer: Total length = $8a$; $\theta = \pi/3$ bisects the upper half (each part = $2a$).

Problem 6. Find the length of the catenary $y = c \cosh(x/c)$ from the vertex $(0, c)$ to any point (x, y) .

$$dy/dx = \sinh(x/c). \, 1 + \sinh^2(x/c) = \cosh^2(x/c).$$

$$s = \int_0^x \cosh(x/c) \, dx = c \sinh(x/c).$$



Answer: $s = c \sinh(x/c)$.

Problem 7. Find the surface area of the sphere formed by rotating the circle $x^2 + y^2 = a^2$ about the x-axis.

$$y = \sqrt{(a^2 - x^2)}, \quad dy/dx = -x/\sqrt{(a^2 - x^2)}. \quad 1 + (dy/dx)^2 = a^2/(a^2 - x^2), \quad \text{so } \sqrt{(1 + (dy/dx)^2)} = a/y.$$

$$S = \int_{-a}^a 2\pi y (a/y) dx = 2\pi a \int_{-a}^a dx = 2\pi a \times 2a.$$

Answer: $S = 4\pi a^2$ square units.

Problem 8. Find the surface area of the solid generated by revolving the arc of the parabola $y^2 = 4ax$ from $x = 0$ to $x = 3a$ about the x-axis.

$$y = 2\sqrt{ax}, \quad dy/dx = \sqrt{a/x}. \quad 1 + (dy/dx)^2 = (x + a)/x.$$

$$2\pi y ds = 2\pi \cdot 2\sqrt{ax} \cdot \sqrt{(x + a)/x} dx = 4\pi\sqrt{a} \cdot \sqrt{(x + a)} dx.$$

$$S = 4\pi\sqrt{a} \int_0^{3a} (x + a)^{1/2} dx = 4\pi\sqrt{a} \cdot (2/3)[(x + a)^{3/2}]_0^{3a} = (8\pi\sqrt{a}/3)[(4a)^{3/2} - a^{3/2}] = (8\pi\sqrt{a}/3)(8a^{3/2} - a^{3/2}).$$

Answer: $S = (56/3) \pi a^2$ square units.

Problem 9. Find the surface area of the solid generated by rotating the cardioid $r = a(1 + \cos \theta)$ about the initial line.

$$ds = 2a \cos(\theta/2) d\theta; \quad \text{radius of revolution} = r \sin \theta = a(1 + \cos \theta) \sin \theta = 2a \cos^2(\theta/2) \cdot 2 \sin(\theta/2) \cos(\theta/2) = 4a \sin(\theta/2) \cos^3(\theta/2).$$

$$S = \int_0^\pi 2\pi \cdot 4a \sin(\theta/2) \cos^3(\theta/2) \cdot 2a \cos(\theta/2) d\theta = 16\pi a^2 \int_0^\pi \sin(\theta/2) \cos^4(\theta/2) d\theta.$$

$$\text{Put } t = \theta/2, \quad d\theta = 2dt: \quad S = 32\pi a^2 \int_0^{\pi/2} \sin t \cos^4 t dt = 32\pi a^2 [-\cos^5 t/5]_0^{\pi/2} = 32\pi a^2/5.$$

Answer: $S = (32/5) \pi a^2$ square units.

Problem 10. Find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the x-axis (prolate spheroid).

$$y^2 = b^2(1 - x^2/a^2). \quad V = \int_{-a}^a \pi y^2 dx = 2\pi b^2 \int_0^a (1 - x^2/a^2) dx.$$

$$= 2\pi b^2 [x - x^3/(3a^2)]_0^a = 2\pi b^2 (a - a/3) = 2\pi b^2 (2a/3).$$

Answer: $V = (4/3) \pi a b^2$ cubic units. (Put $a = b$ to get the sphere $(4/3)\pi a^3$.)

Problem 11. Find the volume of a right circular cone of height h and base radius r using integration.

The cone is formed by rotating the line $y = (r/h)x$, $0 \leq x \leq h$, about the x-axis.

$$V = \int_0^h \pi (r^2/h^2) x^2 dx = \pi (r^2/h^2)(h^3/3).$$

Answer: $V = (1/3) \pi r^2 h$.

Problem 12. The area bounded by $y = x^2$ and $y = x$ is rotated about the x-axis. Find the volume generated.

The curves meet at $x = 0$ and $x = 1$. On $[0, 1]$, $x \geq x^2$, so $y_1 = x$ (outer) and $y_2 = x^2$ (inner).

$$V = \pi \int_0^1 (x^2 - x^4) dx = \pi (1/3 - 1/5) = 2\pi/15.$$



Answer: $V = 2\pi/15$ cubic units.

Problem 13. Using the shell method, find the volume generated when the region under $y = x^2$, $0 \leq x \leq 2$, is rotated about the y-axis.

$$V = \int_0^2 2\pi x \cdot x^2 dx = 2\pi [x^4/4]_0^2 = 2\pi \times 4.$$

Answer: $V = 8\pi$ cubic units.

Problem 14. Find the volume of the solid formed by revolving one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ about its base (x-axis).

$$V = \int \pi y^2 dx = \int_0^{2\pi} \pi a^2(1 - \cos \theta)^2 \cdot a(1 - \cos \theta) d\theta = \pi a^3 \int_0^{2\pi} (1 - \cos \theta)^3 d\theta.$$

$$1 - \cos \theta = 2 \sin^2(\theta/2), \text{ so } (1 - \cos \theta)^3 = 8 \sin^6(\theta/2).$$

$$V = 8\pi a^3 \int_0^{2\pi} \sin^6(\theta/2) d\theta. \text{ Put } t = \theta/2: V = 16\pi a^3 \int_0^{\pi} \sin^6 t dt = 32\pi a^3 \int_0^{\pi/2} \sin^6 t dt.$$

$$\text{By Wallis: } \int_0^{\pi/2} \sin^6 t dt = (5 \cdot 3 \cdot 1)/(6 \cdot 4 \cdot 2) \cdot \pi/2 = 5\pi/32.$$

Answer: $V = 32\pi a^3 \times 5\pi/32 = 5\pi^2 a^3$ cubic units.

Problem 15. Test the convergence of $\int_0^{\infty} e^{-2x} dx$ and evaluate it if it converges.

$$\int_0^t e^{-2x} dx = [-e^{-2x}/2]_0^t = (1 - e^{-2t})/2.$$

$$\text{As } t \rightarrow \infty, e^{-2t} \rightarrow 0, \text{ so the limit is } 1/2.$$

Answer: Converges; value = $1/2$.

Problem 16. Examine $\int_1^{\infty} dx/x$ and $\int_1^{\infty} dx/x^2$.

$$\int_1^t dx/x = \log t \rightarrow \infty \text{ as } t \rightarrow \infty. \text{ So it diverges.}$$

$$\int_1^t dx/x^2 = 1 - 1/t \rightarrow 1 \text{ as } t \rightarrow \infty. \text{ So it converges to } 1.$$

Answer: $\int dx/x$ diverges ($p = 1$); $\int dx/x^2$ converges to 1 ($p = 2 > 1$).

Problem 17. Evaluate $\int_0^1 dx/\sqrt{1-x^2}$ and state its type.

The integrand $\rightarrow \infty$ as $x \rightarrow 1$, so it is improper of Type II.

$$= \lim_{\varepsilon \rightarrow 0} \int_0^{1-\varepsilon} dx/\sqrt{1-x^2} = \lim_{\varepsilon \rightarrow 0} \sin^{-1}(1 - \varepsilon) = \pi/2.$$

Answer: Converges; value = $\pi/2$.

Problem 18. Evaluate $\int_{-\infty}^{\infty} dx/(1+x^2)$.

$$\text{Split at } 0: \text{ each half equals } \lim_{t \rightarrow \infty} \tan^{-1} t = \pi/2.$$

Answer: $\pi/2 + \pi/2 = \pi$ (converges).

Problem 19. Test the convergence of $\int_1^{\infty} dx/(x^2 + 5x + 1)$.

$$\text{For } x \geq 1, x^2 + 5x + 1 > x^2, \text{ so } 1/(x^2 + 5x + 1) < 1/x^2.$$

$$\int_1^{\infty} dx/x^2 \text{ converges } (p = 2 > 1). \text{ By the comparison test, the given integral converges.}$$

Answer: Converges.



Problem 20. Test the convergence of $\int_1^\infty (x+1)/(x^2 \sqrt{x}) dx$ using limit comparison.

For large x , $(x+1)/x^{5/2}$ behaves like $x/x^{5/2} = 1/x^{3/2}$. Take $g(x) = 1/x^{3/2}$.

$\lim f/g = \lim (x+1)/x = 1$, which is finite and non-zero.

$\int_1^\infty dx/x^{3/2}$ converges ($p = 3/2 > 1$).

Answer: The given integral converges.

Problem 21. Evaluate $\Gamma(6)$, $\Gamma(7/2)$ and $\Gamma(-3/2)$.

$$\Gamma(6) = 5! = 120.$$

$$\Gamma(7/2) = (5/2)(3/2)(1/2)\Gamma(1/2) = (15/8)\sqrt{\pi}.$$

$$\Gamma(-1/2) = \Gamma(1/2)/(-1/2) = -2\sqrt{\pi}; \Gamma(-3/2) = \Gamma(-1/2)/(-3/2) = (-2\sqrt{\pi})(-2/3) = 4\sqrt{\pi}/3.$$

Answer: $\Gamma(6) = 120$; $\Gamma(7/2) = 15\sqrt{\pi}/8$; $\Gamma(-3/2) = 4\sqrt{\pi}/3$.

Problem 22. Evaluate $\int_0^\infty x^4 e^{-x} dx$ and $\int_0^\infty x^6 e^{-2x} dx$.

$$\int_0^\infty x^4 e^{-x} dx = \Gamma(5) = 4! = 24.$$

$$\int_0^\infty x^6 e^{-2x} dx = \Gamma(7)/2^7 = 720/128 = 45/8.$$

Answer: 24 and 45/8.

Problem 23. Evaluate $\int_0^\infty \sqrt{x} e^{-x^3} dx$.

$$\text{Put } x^3 = t, \text{ so } x = t^{1/3}, dx = (1/3) t^{-2/3} dt, \sqrt{x} = t^{1/6}.$$

$$\text{Integral} = (1/3) \int_0^\infty t^{1/6} t^{-2/3} e^{-t} dt = (1/3) \int_0^\infty t^{-1/2} e^{-t} dt = (1/3) \Gamma(1/2).$$

Answer: $\sqrt{\pi}/3$.

Problem 24. Evaluate $\int_0^\infty e^{-x^2} dx$ using Gamma function.

$$\text{Put } x^2 = t, x = \sqrt{t}, dx = dt/(2\sqrt{t}).$$

$$\text{Integral} = (1/2) \int_0^\infty t^{-1/2} e^{-t} dt = (1/2) \Gamma(1/2).$$

Answer: $\sqrt{\pi}/2$.

Problem 25. Evaluate $B(3, 4)$ and $B(1/2, 1/2)$.

$$B(3, 4) = \Gamma(3)\Gamma(4)/\Gamma(7) = (2 \times 6)/720 = 12/720 = 1/60.$$

$$B(1/2, 1/2) = \Gamma(1/2)^2/\Gamma(1) = \pi.$$

Answer: $B(3, 4) = 1/60$; $B(1/2, 1/2) = \pi$.

Problem 26. Evaluate $\int_0^1 x^4 (1-x)^3 dx$.

$$\text{This is } B(5, 4) = \Gamma(5)\Gamma(4)/\Gamma(9) = (24 \times 6)/40320 = 144/40320.$$

Answer: 1/280.

Problem 27. Evaluate $\int_0^{\pi/2} \sin^5 \theta \cos^4 \theta d\theta$ using Beta function.

$$\text{Here } p = 5, q = 4. \text{ Integral} = (1/2) B(3, 5/2) = \Gamma(3)\Gamma(5/2)/(2\Gamma(11/2)).$$

$$\Gamma(3) = 2, \Gamma(5/2) = 3\sqrt{\pi}/4, \Gamma(11/2) = (9/2)(7/2)(5/2)\Gamma(5/2) = (315/8)(3\sqrt{\pi}/4).$$

$$\text{Integral} = 2 \times (3\sqrt{\pi}/4) / (2 \times (315/8)(3\sqrt{\pi}/4)) = 1/(315/8) = 8/315.$$



Answer: 8/315. (Check with Wallis: $(4 \cdot 2)(3 \cdot 1)/(9 \cdot 7 \cdot 5 \cdot 3 \cdot 1) = 24/945 = 8/315$.)

Problem 28. Evaluate $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$.

$\sqrt{\tan \theta} = \sin^{1/2} \theta \cos^{-1/2} \theta$. So $p = 1/2$, $q = -1/2$.

Integral = $(1/2) B(3/4, 1/4) = (1/2) \Gamma(3/4)\Gamma(1/4)/\Gamma(1)$.

By the reflection formula, $\Gamma(1/4)\Gamma(3/4) = \pi/\sin(\pi/4) = \pi\sqrt{2}$.

Answer: $\pi/\sqrt{2}$.

Problem 29. Evaluate $\int_0^1 x^2 (1 - x^2)^{3/2} dx$.

Put $x = \sin \theta$, $dx = \cos \theta d\theta$. Integral = $\int_0^{\pi/2} \sin^2 \theta \cos^3 \theta \cdot \cos \theta d\theta = \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta$.

= $\Gamma(3/2)\Gamma(5/2)/(2\Gamma(4)) = (\sqrt{\pi}/2)(3\sqrt{\pi}/4)/(2 \times 6) = (3\pi/8)/12$.

Answer: $\pi/32$.

Problem 30. Evaluate $\int_0^\infty x^2 dx/(1 + x)^6$ using the second form of Beta.

Compare with $x^{m-1}/(1 + x)^{m+n}$: $m - 1 = 2$ gives $m = 3$; $m + n = 6$ gives $n = 3$.

Integral = $B(3, 3) = \Gamma(3)\Gamma(3)/\Gamma(6) = 4/120$.

Answer: 1/30.

Problem 31. Prove that $\int_0^1 x^m (1 - x^n)^p dx = (1/n) B((m + 1)/n, p + 1)$, and hence evaluate $\int_0^1 x^3 (1 - x^2)^{5/2} dx$.

Put $x^n = t$, $x = t^{1/n}$, $dx = (1/n) t^{1/n - 1} dt$.

Integral = $(1/n) \int_0^1 t^{m/n} t^{1/n - 1} (1 - t)^p dt = (1/n) \int_0^1 t^{(m+1)/n - 1} (1 - t)^p dt = (1/n) B((m + 1)/n, p + 1)$.

For $m = 3$, $n = 2$, $p = 5/2$: $(1/2) B(2, 7/2) = (1/2) \Gamma(2)\Gamma(7/2)/\Gamma(11/2) = (1/2) \times 1/((9/2)(7/2)) = (1/2)(4/63)$.

Answer: 2/63.

Problem 32. Show that $B(m, n) = B(m + 1, n) + B(m, n + 1)$.

RHS = $\Gamma(m+1)\Gamma(n)/\Gamma(m+n+1) + \Gamma(m)\Gamma(n+1)/\Gamma(m+n+1) = [m\Gamma(m)\Gamma(n) + n\Gamma(m)\Gamma(n)]/[(m + n)\Gamma(m + n)]$.

= $(m + n)\Gamma(m)\Gamma(n)/[(m + n)\Gamma(m + n)] = \Gamma(m)\Gamma(n)/\Gamma(m + n) = B(m, n)$.

Answer: Proved.

1.17 Practice problems with solutions: Module I

Q1. Find the length of $y = x^{3/2}$ from $x = 0$ to $x = 4$.

$y' = (3/2)x^{1/2}$, $1 + y'^2 = 1 + 9x/4$.

$s = \int_0^4 (1 + 9x/4)^{1/2} dx = (4/9)(2/3)[(1 + 9x/4)^{3/2}]_0^4 = (8/27)(10^{3/2} - 1)$.

Answer: $s = (8/27)(10\sqrt{10} - 1) \approx 9.07$.



Q2. Find the length of the arc of $y^2 = 4ax$ from the vertex to one end of the latus rectum.

The end of the latus rectum is $(a, 2a)$. Use $x = y^2/(4a)$, $dx/dy = y/(2a)$, $0 \leq y \leq 2a$.

$$s = \int_0^{2a} \sqrt{1 + y^2/(4a^2)} dy. \text{ Put } y = 2at: s = 2a \int_0^1 \sqrt{1 + t^2} dt = 2a \left[(t/2)\sqrt{1 + t^2} + (1/2) \log(t + \sqrt{1 + t^2}) \right]_0^1.$$

Answer: $s = a[\sqrt{2} + \log(1 + \sqrt{2})]$.

Q3. Find the perimeter of the circle $r = 2a \cos \theta$.

$r' = -2a \sin \theta$, $r^2 + r'^2 = 4a^2$. $ds/d\theta = 2a$. The full circle is traced for $-\pi/2 \leq \theta \leq \pi/2$.

Answer: $s = 2a \times \pi = 2\pi a$.

Q4. Find the length of the spiral $r = e^\theta$ from $\theta = 0$ to $\theta = 2\pi$.

$r' = e^\theta$, $r^2 + r'^2 = 2e^{2\theta}$, $ds/d\theta = \sqrt{2} e^\theta$.

Answer: $s = \sqrt{2} (e^{2\pi} - 1)$.

Q5. Find the surface area generated by revolving $y = x^3$, $0 \leq x \leq 1$, about the x-axis.

$S = \int_0^1 2\pi x^3 \sqrt{1 + 9x^4} dx$. Put $u = 1 + 9x^4$, $du = 36x^3 dx$.

$$S = (2\pi/36) \int_1^{10} u^{1/2} du = (\pi/18)(2/3)(10^{3/2} - 1).$$

Answer: $S = (\pi/27)(10\sqrt{10} - 1) \approx 3.56$.

Q6. Find the volume generated by rotating $y = \sin x$, $0 \leq x \leq \pi$, about the x-axis.

$$V = \pi \int_0^\pi \sin^2 x dx = \pi \times \pi/2.$$

Answer: $V = \pi^2/2$.

Q7. Find the volume of the solid generated by revolving the region bounded by $y^2 = 4ax$ and $x = a$ about the x-axis.

$$V = \pi \int_0^a 4ax dx = 4\pi a \times a^2/2.$$

Answer: $V = 2\pi a^3$.

Q8. Find the volume of a sphere of radius a using the polar formula.

The semicircular region $r \leq a$, $0 \leq \theta \leq \pi$, rotated about the initial line gives a sphere.

$$V = \int_0^\pi (2\pi/3) a^3 \sin \theta d\theta = (2\pi/3) a^3 \times 2.$$

Answer: $V = (4/3)\pi a^3$.

Q9. Evaluate $\int_0^\infty x e^{-x} dx$.

$$= \Gamma(2) = 1! = 1.$$

Answer: 1.

Q10. Evaluate $\int_0^1 \log x dx$.

Improper at 0. $\int_\epsilon^1 \log x dx = [x \log x - x]_\epsilon^1 = -1 - \epsilon \log \epsilon + \epsilon$. As $\epsilon \rightarrow 0$, $\epsilon \log \epsilon \rightarrow 0$.



Answer: -1 (converges).

Q11. Evaluate $\int_0^\infty \frac{dx}{(1+x)^2}$.

$$= \lim_{t \rightarrow \infty} [-1/(1+x)]_0^t = \lim (1 - 1/(1+t)) = 1.$$

Answer: 1.

Q12. Does $\int_0^1 \frac{dx}{x^2}$ converge?

$$\int_\varepsilon^1 \frac{dx}{x^2} = 1/\varepsilon - 1 \rightarrow \infty \text{ as } \varepsilon \rightarrow 0.$$

Answer: Diverges ($p = 2 \geq 1$).

Q13. Evaluate $\int_0^\infty x^{3/2} e^{-x} dx$.

$$= \Gamma(5/2) = (3/2)(1/2)\sqrt{\pi}.$$

Answer: $3\sqrt{\pi}/4$.

Q14. Evaluate $\int_0^1 x^5(1-x)^2 dx$.

$$= B(6, 3) = \Gamma(6)\Gamma(3)/\Gamma(9) = (120 \times 2)/40320.$$

Answer: $1/168$.

Q15. Evaluate $\int_0^{\pi/2} \sin^6 \theta d\theta$ using the Beta function.

$$p = 6, q = 0: (1/2)B(7/2, 1/2) = \Gamma(7/2)\Gamma(1/2)/(2\Gamma(4)) = (15\sqrt{\pi}/8)(\sqrt{\pi})/12.$$

Answer: $15\pi/96 = 5\pi/32$.

Q16. Show that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} \times \int_0^{\pi/2} \sqrt{\sin \theta} d\theta = \pi$.

See Model Paper, Part B: the product equals $\pi\Gamma(1/4)/(4\Gamma(5/4)) = \pi$ since $\Gamma(5/4) = \Gamma(1/4)/4$.

Answer: π .

Q17. Evaluate $\int_0^2 x^2 \sqrt{2-x} dx$.

$$\text{Put } x = 2t, dx = 2dt: \int_0^1 4t^2 \sqrt{2} \sqrt{1-t} 2 dt = 8\sqrt{2} B(3, 3/2) = 8\sqrt{2} \times 16/105.$$

Answer: $128\sqrt{2}/105$.

Q18. Show that $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx = 0$.

$$\text{Split: } \int x^8/(1+x)^{24} dx - \int x^{14}/(1+x)^{24} dx = B(9, 15) - B(15, 9).$$

Answer: 0, since Beta is symmetric.

1.18 MCQs: Module I

1. The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

- (a) $\int \pi y^2 dx$ (b) $\int \sqrt{1+y'} dx$ (c) $\int (1+(y')^2) dx$ (d) $\int \sqrt{1+(y')^2} dx$
☐ (d) $-ds^2 = dx^2 + dy^2$.

2. For a polar curve $r = f(\theta)$, $ds/d\theta$ equals

- (a) $\sqrt{r^2 + (dr/d\theta)^2}$ (b) $r dr/d\theta$ (c) $\sqrt{1 + (dr/d\theta)^2}$ (d) $\sqrt{1 + r^2}$
☐ (a) — Arc element in polar coordinates.



3. The perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is
(a) $8a$ (b) $4a$ (c) $6a$ (d) $2\pi a$
□ (c) — Four quadrants $\times 3a/2$.
4. The total length of the cardioid $r = a(1 + \cos \theta)$ is
(a) $6a$ (b) $8a$ (c) $4a$ (d) $16a$
□ (b) — $2 \times \int_0^\pi 2a \cos(\theta/2) d\theta$.
5. The length of one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ is
(a) $6a$ (b) $2\pi a$ (c) $4a$ (d) $8a$
□ (d) — $\int_0^{2\pi} 2a \sin(\theta/2) d\theta$.
6. Length of the arc of $y = \log \sec x$ from 0 to $\pi/4$ is
(a) $\pi/4$ (b) $\log(1 + \sqrt{2})$ (c) $\log 2$ (d) $\sqrt{2}$
□ (b) — $\int \sec x dx = \log(\sec x + \tan x)$.
7. Surface area generated by revolving $y = f(x)$ about the x-axis is
(a) $\int 2\pi y ds$ (b) $\int 2\pi y dx$ (c) $\int \pi y^2 dx$ (d) $\int 2\pi x ds$
□ (a) — Band of radius y and width ds .
8. The surface area of a sphere of radius a is
(a) $(4/3)\pi a^3$ (b) $4\pi a^2$ (c) $2\pi a^2$ (d) πa^2
□ (b) — Rotate a semicircle.
9. Volume generated by revolving the region under $y = f(x)$ from a to b about the x-axis is
(a) $\int \pi x^2 dy$ (b) $\int 2\pi y dx$ (c) $\int y dx$ (d) $\int \pi y^2 dx$
□ (d) — Disc method.
10. The shell-method volume for rotation about the y-axis is
(a) $\int 2\pi x y dx$ (b) $\int \pi y^2 dx$ (c) $\int 2\pi y dx$ (d) $\int \pi x^2 dy$
□ (a) — Shell of radius x and height y .
11. Volume of the solid generated by revolving $y = x$, $0 \leq x \leq 1$, about x-axis is
(a) π (b) $\pi/2$ (c) $\pi/3$ (d) $2\pi/3$
□ (c) — $\pi \int x^2 dx = \pi/3$.
12. Volume of the prolate spheroid (ellipse $x^2/a^2 + y^2/b^2 = 1$ rotated about x-axis) is
(a) $(4/3)\pi ab^2$ (b) πab^2 (c) $(4/3)\pi a^2 b$ (d) $(4/3)\pi ab$
□ (a) — $\pi \int y^2 dx$ from $-a$ to a .
13. Volume of the torus formed by rotating a circle of radius a about a line at distance b ($b > a$) from its centre is
(a) $4\pi^2 ab$ (b) $2\pi^2 a^2 b$ (c) $\pi^2 a^2 b$ (d) $2\pi a^2 b$
□ (b) — Pappus: $\pi a^2 \times 2\pi b$.
14. Volume generated by rotating one arch of the cycloid about its base is
(a) $3\pi^2 a^3$ (b) $5\pi a^3$ (c) $5\pi^2 a^3$ (d) $6\pi^2 a^3$
□ (c) — Standard result.
15. $\int_1^\infty dx/x^p$ converges when
(a) $p > 1$ (b) $p \leq 1$ (c) $p = 1$ (d) $p < 1$
□ (a) — p-test for infinite range.
16. $\int_0^1 dx/x^p$ converges when
(a) $p \geq 1$ (b) $p > 1$ (c) for all p (d) $p < 1$
□ (d) — p-test for finite range with singularity at 0.
17. $\int_1^\infty dx/x$ is
(a) 1 (b) 0 (c) divergent (d) $\log 2$
□ (c) — $\log t \rightarrow \infty$.



- 18.** $\int_0^{\infty} e^{-3x} dx$ equals
(a) 3 (b) 0 (c) $1/3$ (d) ∞
□ (c) — $\int e^{-ax} = 1/a$.
- 19.** $\int_0^1 dx/\sqrt{x}$ equals
(a) $1/2$ (b) 2 (c) 1 (d) divergent
□ (b) — $[2\sqrt{x}]_0^1 = 2$; $p = 1/2 < 1$.
- 20.** $\int_{-\infty}^{\infty} dx/(1 + x^2)$ equals
(a) π (b) 2π (c) $\pi/2$ (d) divergent
□ (a) — $\tan^{-1}x$ from $-\infty$ to ∞ .
- 21.** $\int_0^1 \log x dx$ equals
(a) 1 (b) 0 (c) divergent (d) -1
□ (d) — $[x \log x - x]_0^1 = -1$.
- 22.** An integral $\int_a^b f(x) dx$ where $f \rightarrow \infty$ at some point in $[a, b]$ is called
(a) proper (b) definite (c) improper of the second kind (d) improper of the first kind
□ (c) — Type II = unbounded integrand.
- 23.** $\Gamma(n)$ is defined for
(a) $n > 0$ (as an integral) (b) $n < 0$ only (c) all real n (d) integers only
□ (a) — The integral converges for $n > 0$.
- 24.** $\Gamma(1)$ equals
(a) 0 (b) 1 (c) $\sqrt{\pi}$ (d) ∞
□ (b) — $\int e^{-x} dx = 1$.
- 25.** $\Gamma(n + 1)$ equals
(a) $\Gamma(n)/n$ (b) $\Gamma(n) + 1$ (c) $(n + 1)\Gamma(n)$ (d) $n\Gamma(n)$
□ (d) — Integration by parts.
- 26.** $\Gamma(1/2)$ equals
(a) $\sqrt{\pi}$ (b) 1 (c) π (d) $\pi/2$
□ (a) — Reduces to $2\int e^{-t^2} dt$.
- 27.** $\Gamma(5)$ equals
(a) 120 (b) 5 (c) 24 (d) 20
□ (c) — $\Gamma(5) = 4! = 24$.
- 28.** $\Gamma(3/2)$ equals
(a) $\sqrt{\pi}$ (b) $3\sqrt{\pi}/4$ (c) $\pi/2$ (d) $\sqrt{\pi}/2$
□ (d) — $(1/2)\Gamma(1/2)$.
- 29.** $\Gamma(5/2)$ equals
(a) $3\sqrt{\pi}/4$ (b) $3\sqrt{\pi}/2$ (c) $15\sqrt{\pi}/8$ (d) $\sqrt{\pi}/2$
□ (a) — $(3/2)(1/2)\sqrt{\pi}$.
- 30.** $\Gamma(-1/2)$ equals
(a) not defined (b) $\sqrt{\pi}/2$ (c) $2\sqrt{\pi}$ (d) $-2\sqrt{\pi}$
□ (d) — $\Gamma(1/2)/(-1/2)$.
- 31.** $\Gamma(1/4)\Gamma(3/4)$ equals
(a) $\pi\sqrt{2}$ (b) $\pi/\sqrt{2}$ (c) π (d) 2π
□ (a) — $\pi/\sin(\pi/4)$.
- 32.** $\Gamma(n)\Gamma(1 - n)$ equals
(a) $\pi/\cos n\pi$ (b) $\sin n\pi/\pi$ (c) $n\pi$ (d) $\pi/\sin n\pi$
□ (d) — Reflection formula, $0 < n < 1$.



- 33.** $\int_0^{\infty} x^3 e^{-x} dx$ equals
(a) 3 (b) 2 (c) 6 (d) 24
□ (c) — $\Gamma(4) = 3! = 6$.
- 34.** $\int_0^{\infty} e^{-x^2} dx$ equals
(a) 1 (b) $\pi/2$ (c) $\sqrt{\pi}/2$ (d) $\sqrt{\pi}$
□ (c) — $(1/2)\Gamma(1/2)$.
- 35.** $\int_0^{\infty} x^{n-1} e^{-kx} dx$ equals
(a) $k\Gamma(n)$ (b) $\Gamma(n)k^n$ (c) $\Gamma(n)/k$ (d) $\Gamma(n)/k^n$
□ (d) — Put $kx = t$.
- 36.** $B(m, n)$ in terms of Gamma is
(a) $\Gamma(m)\Gamma(n)$ (b) $\Gamma(mn)$ (c) $\Gamma(m)\Gamma(n)/\Gamma(m+n)$ (d) $\Gamma(m+n)/\Gamma(m)\Gamma(n)$
□ (c) — Standard relation.
- 37.** $B(m, n) - B(n, m)$ equals
(a) $m - n$ (b) mn (c) 1 (d) 0
□ (d) — Beta is symmetric.
- 38.** $B(2, 3)$ equals
(a) $1/12$ (b) $1/6$ (c) $1/2$ (d) $1/24$
□ (a) — $1! \cdot 2! / 4! = 2/24$.
- 39.** $B(1/2, 1/2)$ equals
(a) $\pi/2$ (b) $\sqrt{\pi}$ (c) 1 (d) π
□ (d) — $\Gamma(1/2)^2 / \Gamma(1)$.
- 40.** $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$ equals
(a) $B(m, n)$ (b) $(1/2)B(m, n)$ (c) $2B(m, n)$ (d) $B(2m, 2n)$
□ (b) — Trigonometric form.
- 41.** $\int_0^1 x^3(1-x)^4 dx$ equals
(a) $B(3, 4)$ (b) $B(3, 5)$ (c) $B(4, 4)$ (d) $B(4, 5)$
□ (d) — Exponents are $m-1$ and $n-1$.
- 42.** $\int_0^{\infty} x^{m-1}/(1+x)^{m+n} dx$ equals
(a) $\Gamma(m)$ (b) $B(m, n)$ (c) $\Gamma(m+n)$ (d) $B(m, n)/2$
□ (b) — Second form of Beta.
- 43.** $\int_0^{\pi/2} \sin^4 x dx$ equals
(a) $\pi/4$ (b) $3\pi/16$ (c) $3/8$ (d) $3\pi/8$
□ (b) — Wallis: $(3 \cdot 1)/(4 \cdot 2) \cdot \pi/2$.
- 44.** $\int_0^{\pi/2} \cos^5 x dx$ equals
(a) $16/15$ (b) $8/15$ (c) $5\pi/32$ (d) $4/15$
□ (b) — Wallis for odd n : $(4 \cdot 2)/(5 \cdot 3)$.
- 45.** $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$ equals
(a) $\sqrt{\pi}$ (b) $\pi/\sqrt{2}$ (c) $\pi\sqrt{2}$ (d) $\pi/2$
□ (b) — $(1/2)\Gamma(3/4)\Gamma(1/4)$.
- 46.** The value of $\int_0^{\infty} dx/(1+x)^2$ is
(a) 1 (b) $1/2$ (c) ∞ (d) 2
□ (a) — $[-1/(1+x)]_0^{\infty} = 1$.
- 47.** Using Pappus's theorem, the volume generated is area times
(a) the centroid distance only (b) the radius (c) the distance travelled by the centroid (d) the perimeter
□ (c) — $V = A \times 2\pi\bar{y}$.



48. $\Gamma(0)$ is

(a) 0 (b) not defined (infinite) (c) 1 (d) $\sqrt{\pi}$

□ (b) — $\Gamma(n) = \Gamma(n + 1)/n$ with $n \rightarrow 0$.

49. The duplication formula states $\Gamma(m)\Gamma(m + 1/2)$ equals

(a) $\Gamma(2m)/2^m$ (b) $\pi\Gamma(2m)$ (c) $2^{2m}\Gamma(m)$ (d) $\sqrt{\pi} \Gamma(2m)/2^{2m-1}$

□ (d) — Legendre duplication formula.

50. $\int_0^1 x^m (\log x)^n dx$ for positive integer n equals

(a) $n!/(m + 1)^n$ (b) $(-1)^n/(m + 1)$ (c) 0 (d) $(-1)^n n!/(m + 1)^{n+1}$

□ (d) — Put $x = e^{-t}$.

Module II: Single-variable Calculus (Differentiation)

This module studies what the derivative tells us about a function. Rolle's theorem and the Mean Value Theorems connect the value of a function at two points with its derivative somewhere in between. From them we get tests for increasing/decreasing functions, extreme values, and the Taylor series that approximates a function by a polynomial.

2.1 Continuity and differentiability (quick review)

- f is **continuous at c** if $\lim_{x \rightarrow c} f(x) = f(c)$. f is continuous on $[a, b]$ if it is continuous at every interior point and one-sided continuous at a and b .
- f is **differentiable at c** if $f'(c) = \lim_{h \rightarrow 0} [f(c+h) - f(c)]/h$ exists (finite).
- Differentiability implies continuity, but not conversely. Example: $f(x) = |x|$ is continuous at 0 but not differentiable there (left derivative -1 , right derivative $+1$).
- Polynomials, $\sin x$, $\cos x$, e^x are continuous and differentiable everywhere; $\log x$ for $x > 0$; $\tan x$ except at odd multiples of $\pi/2$.
- A continuous function on a closed interval $[a, b]$ is bounded and attains its maximum and minimum (Extreme Value Theorem). It also takes every value between $f(a)$ and $f(b)$ (Intermediate Value Theorem).

2.2 Rolle's theorem

Statement. Let f be a function such that

- (i) f is continuous on the closed interval $[a, b]$,
- (ii) f is differentiable on the open interval (a, b) , and
- (iii) $f(a) = f(b)$.

Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Geometrical meaning. If a smooth curve starts and ends at the same height, then somewhere between the ends the tangent to the curve is horizontal (parallel to the x -axis).

Algebraic meaning. Between two roots of $f(x) = 0$ there is at least one root of $f'(x) = 0$ (put $f(a) = f(b) = 0$).

Proof (outline). f is continuous on $[a, b]$, so it attains a maximum M and a minimum m . If $M = m$, f is constant and $f'(x) = 0$ everywhere. Otherwise at least one of M, m differs from $f(a) = f(b)$, so it is attained at an interior point c . At an interior maximum or minimum of a differentiable function, $f'(c) = 0$ (the left and right difference quotients have opposite signs, so the limit must be 0).

When Rolle's theorem does not apply (common exam questions):

- $f(x) = |x|$ on $[-1, 1]$: $f(-1) = f(1) = 1$ but f is not differentiable at 0 . There is no c with $f'(c) = 0$.
- $f(x) = \tan x$ on $[0, \pi]$: $f(0) = f(\pi) = 0$ but f is discontinuous at $\pi/2$.
- $f(x) = x$ on $[0, 1]$: continuous and differentiable but $f(0) \neq f(1)$.
- $f(x) = 1/x^2$ on $[-1, 1]$: discontinuous at 0 .
- $f(x) = x^{2/3}$ on $[-1, 1]$: $f'(x) = (2/3)x^{-1/3}$ does not exist at $x = 0$.

2.3 Lagrange's Mean Value Theorem (LMVT)

Statement. If f is (i) continuous on $[a, b]$ and (ii) differentiable on (a, b) , then there exists at least one c in (a, b) such that

$$f'(c) = [f(b) - f(a)] / (b - a)$$

Geometrical meaning. Somewhere between $A(a, f(a))$ and $B(b, f(b))$, the tangent to the curve is parallel to the chord AB.

Physical meaning. If you travel 120 km in 2 hours, at some instant your speed was exactly 60 km/h (instantaneous speed = average speed at some instant).

Proof. Define $\varphi(x) = f(x) - f(a) - k(x - a)$, where $k = [f(b) - f(a)]/(b - a)$. Then φ is continuous on $[a, b]$, differentiable on (a, b) , and $\varphi(a) = \varphi(b) = 0$. By Rolle's theorem there is c with $\varphi'(c) = 0$, that is, $f'(c) - k = 0$, so $f'(c) = k$.

Other form. Put $b = a + h$ and $c = a + \theta h$ with $0 < \theta < 1$:

$$f(a + h) = f(a) + h f'(a + \theta h), \quad 0 < \theta < 1$$

Consequences of LMVT (applications):

- If $f'(x) = 0$ for all x in (a, b) , then f is constant on $[a, b]$.
- If $f'(x) = g'(x)$ for all x in (a, b) , then $f(x) = g(x) + \text{constant}$.
- If $f'(x) > 0$ on (a, b) , f is strictly increasing on $[a, b]$; if $f'(x) < 0$, f is strictly decreasing.
- Proving inequalities: e.g. $|\sin b - \sin a| \leq |b - a|$, because $|\cos c| \leq 1$.
- Error estimates: $|f(b) - f(a)| \leq M|b - a|$ where $M = \max |f'|$.

2.4 Cauchy's Mean Value Theorem (CMVT)

Statement. If f and g are continuous on $[a, b]$, differentiable on (a, b) , and $g'(x) \neq 0$ in (a, b) , then there exists c in (a, b) such that

$$[f(b) - f(a)] / [g(b) - g(a)] = f'(c) / g'(c)$$

Taking $g(x) = x$ gives LMVT. CMVT is the basis for L'Hôpital's rule.

Proof. Apply Rolle's theorem to $\varphi(x) = f(x) - k g(x)$, where $k = [f(b) - f(a)]/[g(b) - g(a)]$. Then $\varphi(a) = \varphi(b)$, so $\varphi'(c) = 0$ gives $f'(c) = k g'(c)$.

Standard results for CMVT:

- $f = e^x$, $g = e^{-x}$ on $[a, b]$: $c = (a + b)/2$ (arithmetic mean).
- $f = \sqrt{x}$, $g = 1/\sqrt{x}$ on $[a, b]$: $c = \sqrt{ab}$ (geometric mean).
- $f = 1/x^2$, $g = 1/x$ on $[a, b]$: $c = 2ab/(a + b)$ (harmonic mean).

2.5 Increasing and decreasing functions

$$f'(x) > 0 \text{ on an interval} \Rightarrow f \text{ strictly increasing; } f'(x) < 0 \Rightarrow \text{strictly decreasing; } f'(x) = 0 \Rightarrow \text{constant}$$

Working rule: find $f'(x)$, find where $f'(x) = 0$ or does not exist (critical points), split the real line at these points, and check the sign of f' in each interval.

2.6 Extreme values: first derivative test

Definitions.

- f has a **local (relative) maximum** at c if $f(c) \geq f(x)$ for all x near c . It has a **local minimum** if $f(c) \leq f(x)$ for all x near c .
- The **absolute (global) maximum** on a domain D is the largest value of f on all of D .
- A **critical point** is an interior point c where $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's theorem. If f has a local extreme value at an interior point c and $f'(c)$ exists, then $f'(c) = 0$. (The converse is false: $f(x) = x^3$ has $f'(0) = 0$ but no extremum at 0.)

First derivative test. Let c be a critical point.

- If f' changes sign from $+$ to $-$ as x increases through c , f has a **local maximum** at c .
- If f' changes sign from $-$ to $+$, f has a **local minimum** at c .
- If f' does not change sign, there is no extremum at c (it may be a point of inflection).

Second derivative test (for comparison). If $f'(c) = 0$ and $f''(c) < 0$, local max; if $f''(c) > 0$, local min; if $f''(c) = 0$, the test fails (use the first derivative test or higher derivatives).

Absolute extrema on $[a, b]$ (closed interval method): evaluate f at all critical points in (a, b) and at the end points a and b . The largest value is the absolute max; the smallest is the absolute min.

2.7 Sequences and series: essentials for power series

- An infinite series $\sum a_n$ **converges** if the sequence of partial sums $S_n = a_1 + \dots + a_n$ tends to a finite limit.
- **Geometric series** $\sum r^n$ converges to $1/(1 - r)$ when $|r| < 1$ and diverges when $|r| \geq 1$.
- **p-series** $\sum 1/n^p$ converges iff $p > 1$. The harmonic series $\sum 1/n$ diverges.
- **n-th term test:** if a_n does not tend to 0, the series diverges.
- **Ratio test (D'Alembert):** if $\lim |a_{n+1}/a_n| = L$, the series converges absolutely if $L < 1$, diverges if $L > 1$, and the test fails if $L = 1$.
- **Root test (Cauchy):** if $\lim |a_n|^{1/n} = L$, same conclusions as the ratio test.

2.8 Power series

A **power series** about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots$$

For every power series there is a number R ($0 \leq R \leq \infty$), called the **radius of convergence**, such that the series converges absolutely for $|x - a| < R$ and diverges for $|x - a| > R$. At the end points $x = a \pm R$, it must be checked separately.

$$R = \lim_{n \rightarrow \infty} |c_n / c_{n+1}| \text{ (ratio method) or } 1/R = \lim_{n \rightarrow \infty} |c_n|^{1/n} \text{ (root method)}$$

The **interval of convergence** is $(a - R, a + R)$ together with any end points at which the series converges.

Properties. Inside the interval of convergence, a power series can be differentiated and integrated term by term, and the new series has the same radius of convergence.

Examples:

- $\sum x^n$: $R = 1$, interval $(-1, 1)$.
- $\sum x^n/n$: $R = 1$, interval $[-1, 1)$ (at $x = -1$ alternating harmonic series converges; at $x = 1$ harmonic series diverges).
- $\sum x^n/n!$: $R = \infty$ (converges for all x ; it is e^x).
- $\sum n! x^n$: $R = 0$ (converges only at $x = 0$).

2.9 Taylor's theorem and Taylor series

Taylor's theorem with Lagrange's form of remainder. If f and its first n derivatives are continuous on $[a, a + h]$ and $f^{(n+1)}$ exists on $(a, a + h)$, then

$$f(a + h) = f(a) + h f'(a) + (h^2/2!) f''(a) + \dots + (h^n/n!) f^{(n)}(a) + R_n$$

$$R_n = (h^{n+1}/(n+1)!) f^{(n+1)}(a + \theta h), 0 < \theta < 1 \text{ (Lagrange's form of remainder)}$$

For $n = 0$ this is exactly LMVT. The remainder tells us the error when a function is replaced by its Taylor polynomial.

Taylor series about $x = a$. If $R_n \rightarrow 0$ as $n \rightarrow \infty$,

$$f(x) = f(a) + (x - a) f'(a) + ((x - a)^2/2!) f''(a) + ((x - a)^3/3!) f'''(a) + \dots$$

Maclaurin series is the Taylor series about $a = 0$:

$$f(x) = f(0) + x f'(0) + (x^2/2!) f''(0) + (x^3/3!) f'''(0) + \dots$$

Standard Maclaurin expansions (memorise):

$$e^x = 1 + x + x^2/2! + x^3/3! + \dots \text{ (all } x)$$

$$\sin x = x - x^3/3! + x^5/5! - \dots \text{ (all } x) \quad \cos x = 1 - x^2/2! + x^4/4! - \dots \text{ (all } x)$$

$$\log(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \dots \text{ } (-1 < x \leq 1)$$

$$(1 + x)^n = 1 + nx + n(n-1)x^2/2! + n(n-1)(n-2)x^3/3! + \dots \text{ } (|x| < 1)$$

$$1/(1 - x) = 1 + x + x^2 + x^3 + \dots \text{ } (|x| < 1)$$

$$\tan x = x + x^3/3 + 2x^5/15 + \dots \text{ } (|x| < \pi/2)$$

$$\tan^{-1} x = x - x^3/3 + x^5/5 - \dots \text{ } (|x| \leq 1)$$

$$\sinh x = x + x^3/3! + x^5/5! + \dots \quad \cosh x = 1 + x^2/2! + x^4/4! + \dots$$

Working rule for expansion.

- Find $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$, ... (as many terms as asked).
- Evaluate each at $x = a$ (or $x = 0$ for Maclaurin).
- Substitute into the series formula.
- For products and compositions it is often faster to multiply/substitute known series, e.g. $e^x \sin x$ or $e^{\sin x}$.

Using series for approximations. Taylor polynomials give quick approximations: $\sqrt{1.02} \approx 1 + (1/2)(0.02) - (1/8)(0.02)^2 = 1.00995$. The error is bounded by the next term's size (Lagrange remainder).

Using series for limits. $\lim_{x \rightarrow 0} (\sin x - x)/x^3 = \lim_{x \rightarrow 0} (-x^3/6 + x^5/120 - \dots)/x^3 = -1/6$.

2.10 L'Hôpital's rule (application of Cauchy's MVT)

Cauchy's MVT gives a powerful way to evaluate limits of the indeterminate forms $0/0$ and ∞/∞ .

Statement. If $f(a) = g(a) = 0$ (or both $\rightarrow \infty$), f and g are differentiable near a , and $g'(x) \neq 0$ near a , then

$$\lim_{x \rightarrow a} f(x)/g(x) = \lim_{x \rightarrow a} f'(x)/g'(x), \text{ provided the right-hand limit exists}$$

Proof idea (0/0 case). By CMVT on $[a, x]$: $[f(x) - f(a)]/[g(x) - g(a)] = f'(c)/g'(c)$ with $a < c < x$. Since $f(a) = g(a) = 0$, $f(x)/g(x) = f'(c)/g'(c)$. As $x \rightarrow a$, $c \rightarrow a$.

Other indeterminate forms are converted to $0/0$ or ∞/∞ first:

- $0 \times \infty$: write fg as $f/(1/g)$.

- $\infty - \infty$: combine into a single fraction.
- $1^\infty, 0^0, \infty^0$: take logarithms. If $y = f^g$, then $\log y = g \log f$.

Examples.

- $\lim_{x \rightarrow 0} (\sin x)/x = \lim \cos x/1 = 1$.
- $\lim_{x \rightarrow 0} (1 - \cos x)/x^2 = \lim \sin x/(2x) = 1/2$.
- $\lim_{x \rightarrow \infty} x^2/e^x = \lim 2x/e^x = \lim 2/e^x = 0$.
- $\lim_{x \rightarrow 0^+} x \log x = \lim \log x/(1/x) = \lim (1/x)/(-1/x^2) = \lim (-x) = 0$.
- $\lim_{x \rightarrow 0} (1 + x)^{1/x}$: $\log y = \log(1 + x)/x \rightarrow 1$, so the limit is e .

2.11 Concavity, points of inflection and the second derivative

- f is **concave up** (convex) on an interval if $f''(x) > 0$ there; the curve lies above its tangents and bends upward like a cup.
- f is **concave down** if $f''(x) < 0$; the curve lies below its tangents and bends downward like a cap.
- A **point of inflection** is a point where the concavity changes. At such a point $f'' = 0$ or f'' does not exist, and f'' changes sign.

Example. $f(x) = x^3 - 3x^2 + 1$; $f'(x) = 6x - 6$. For $x < 1$, $f' < 0$ (concave down); for $x > 1$, $f' > 0$ (concave up). So $(1, -1)$ is a point of inflection.

Why the second derivative test works. If $f'(c) = 0$ and $f''(c) > 0$, then f' is increasing near c . So $f' < 0$ just left of c and $f' > 0$ just right of c . By the first derivative test, c is a local minimum. The case $f''(c) < 0$ is similar.

2.12 Proof of Taylor's theorem (outline, for long answers)

Let $F(x) = f(b) - f(x) - (b - x)f'(x) - (b - x)^2 f''(x)/2! - \dots - (b - x)^n f^{(n)}(x)/n! - A(b - x)^{n+1}$, where A is chosen so that $F(a) = 0$. Also $F(b) = 0$ directly. By Rolle's theorem, $F'(c) = 0$ for some $c \in (a, b)$. On differentiating, all terms cancel in pairs except $-(b - x)^n f^{(n+1)}(x)/n! + A(n + 1)(b - x)^n$. Setting this to 0 at $x = c$ gives $A = f^{(n+1)}(c)/(n + 1)!$. Putting $b = a + h$ and $c = a + \theta h$ gives Taylor's formula with Lagrange's remainder.

Other forms of the remainder.

Cauchy's form: $R_n = h^{n+1}(1 - \theta)^n f^{(n+1)}(a + \theta h)/n!$

Integral form: $R_n = (1/n!) \int_a^x (x - t)^n f^{(n+1)}(t) dt$

When does a Taylor series converge to f ? When $R_n \rightarrow 0$ as $n \rightarrow \infty$. For e^x , $\sin x$ and $\cos x$, $|f^{(n+1)}|$ is bounded on any interval and $h^{n+1}/(n + 1)! \rightarrow 0$, so the series converge to f for all x .

2.13 Tests for convergence of series (summary for power series work)

Test	Statement	Best used when
n-th term	If a_n does not $\rightarrow 0$, $\sum a_n$ diverges	Quick first check
Comparison	$0 \leq a_n \leq b_n$: $\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent	Terms like $1/(n^2 + 1)$
Limit comparison	$a_n/b_n \rightarrow L$ ($0 < L < \infty$) = same behaviour	Rational functions of n
Ratio (D'Alembert)	$\lim a_{n+1}/a_n = L$; $L < 1$ conv, $L > 1$ div	Factorials, powers like 2^n , $n!$

Test	Statement	Best used when
Root (Cauchy)	$\lim a_n^{1/n} = L$; $L < 1$ conv, $L > 1$ div	Terms like $(n/(n+1))^{n^2}$
Integral	$\sum f(n)$ and $\int f(x)dx$ behave alike (f positive, decreasing)	$1/(n \log n)$, $1/n^p$
Leibniz (alternating)	$\sum (-1)^n a_n$ converges if a_n decreases to 0	Alternating series

Absolute and conditional convergence. $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ does not, it converges conditionally (e.g. $1 - 1/2 + 1/3 - \dots = \log 2$).

2.14 Physical meaning and engineering uses

- **LMVT in motion:** average velocity over a time interval equals the instantaneous velocity at some instant. Speed-camera systems that measure average speed between two points use exactly this idea.
- **Error bounds:** if $|f'(x)| \leq M$, then $|f(b) - f(a)| \leq M|b - a|$. This bounds how much an output changes when an input changes.
- **Linear approximation:** $f(a + h) \approx f(a) + hf'(a)$ is Taylor's formula with one term. It is used in small-signal circuit analysis and in Newton-Raphson root finding.
- **Series in calculators and computers:** values of $\sin x$, e^x and $\log x$ are computed from polynomial approximations derived from Taylor series.
- **Small-angle approximations:** $\sin \theta \approx \theta$, $\cos \theta \approx 1 - \theta^2/2$ (used in the simple pendulum and optics).

2.15 Common mistakes to avoid in Module II

- Forgetting to check the hypotheses of Rolle/LMVT before finding c .
- Giving a value of c that lies outside the open interval (a, b) . Always state that c lies in the interval.
- Concluding "maximum" from $f'(c) = 0$ alone. $f'(c) = 0$ is necessary but not sufficient.
- In Maclaurin series, forgetting the factorials in the denominators.
- In radius of convergence, forgetting to test the end points separately.

2.16 Solved problems: Module II

Problem 1. Verify Rolle's theorem for $f(x) = x^2 - 5x + 6$ on $[2, 3]$.

f is a polynomial, so it is continuous on $[2, 3]$ and differentiable on $(2, 3)$.

$f(2) = 4 - 10 + 6 = 0$ and $f(3) = 9 - 15 + 6 = 0$, so $f(2) = f(3)$.

$f'(x) = 2x - 5 = 0$ gives $c = 5/2 = 2.5$, which lies in $(2, 3)$.

Answer: Rolle's theorem is verified with $c = 5/2$.

Problem 2. Verify Rolle's theorem for $f(x) = \sin x / e^x$ on $[0, \pi]$.

$\sin x$ and e^x are continuous and differentiable everywhere, and $e^x \neq 0$, so f is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$f(0) = 0$ and $f(\pi) = 0$.

$f'(x) = e^{-x}(\cos x - \sin x) = 0$ gives $\tan x = 1$, so $c = \pi/4$, which lies in $(0, \pi)$.



Answer: Verified; $c = \pi/4$.

Problem 3. Verify Rolle's theorem for $f(x) = x(x + 3)e^{-x/2}$ on $[-3, 0]$.

f is a product of a polynomial and an exponential, hence continuous and differentiable. $f(-3) = 0 = f(0)$.

$$f'(x) = e^{-x/2}[(2x + 3) - (x^2 + 3x)/2] = (e^{-x/2}/2)(-x^2 + x + 6) = -(e^{-x/2}/2)(x - 3)(x + 2).$$

$f'(c) = 0$ gives $c = 3$ or $c = -2$. Only $c = -2$ lies in $(-3, 0)$.

Answer: Verified; $c = -2$.

Problem 4. Verify Rolle's theorem for $f(x) = \log[(x^2 + ab)/(x(a + b))]$ on $[a, b]$, $0 < a < b$.

For x in $[a, b]$ ($x > 0$), the argument of log is positive, so f is continuous and differentiable.

$$f(a) = \log[(a^2 + ab)/(a(a + b))] = \log 1 = 0; f(b) = \log[(b^2 + ab)/(b(a + b))] = \log 1 = 0.$$

$$f(x) = \log(x^2 + ab) - \log x - \log(a + b). f'(x) = 2x/(x^2 + ab) - 1/x = (x^2 - ab)/[x(x^2 + ab)].$$

$f'(c) = 0$ gives $c^2 = ab$, $c = \sqrt{ab}$, which lies between a and b .

Answer: Verified; $c = \sqrt{ab}$ (the geometric mean).

Problem 5. Discuss the applicability of Rolle's theorem to $f(x) = |x|$ on $[-1, 1]$.

f is continuous on $[-1, 1]$ and $f(-1) = f(1) = 1$.

But $f'(0)$ does not exist (left derivative -1 , right derivative 1). So condition (ii) fails.

Indeed $f'(x) = -1$ for $x < 0$ and 1 for $x > 0$, so f' is never 0 .

Answer: Rolle's theorem is not applicable.

Problem 6. Show that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Let $f(x) = x^3 + x - 1$. $f(0) = -1 < 0$ and $f(1) = 1 > 0$. By the Intermediate Value Theorem, there is at least one root in $(0, 1)$.

Suppose there were two roots $a < b$. Then $f(a) = f(b) = 0$, and by Rolle's theorem $f'(c) = 0$ for some c in (a, b) .

But $f'(x) = 3x^2 + 1 > 0$ for all x , so this is impossible.

Answer: The equation has exactly one real root (lying in $(0, 1)$).

Problem 7. Verify Lagrange's Mean Value Theorem for $f(x) = x^3 - 3x^2 + 2x$ on $[0, 1/2]$.

f is a polynomial, hence continuous and differentiable.

$$f(0) = 0, f(1/2) = 1/8 - 3/4 + 1 = 3/8. [f(b) - f(a)]/(b - a) = (3/8)/(1/2) = 3/4.$$

$$f'(x) = 3x^2 - 6x + 2 = 3/4 \Rightarrow 12x^2 - 24x + 5 = 0 \Rightarrow x = [24 \pm \sqrt{576 - 240}]/24 = 1 \pm \sqrt{21}/6.$$

$1 - \sqrt{21}/6 \approx 1 - 0.764 = 0.236$, which lies in $(0, 1/2)$. The other root 1.764 does not.



Answer: Verified; $c = 1 - \sqrt{21/6} \approx 0.236$.

Problem 8. Verify LMVT for $f(x) = \log x$ on $[1, e]$.

$\log x$ is continuous on $[1, e]$ and differentiable on $(1, e)$.

$$[f(e) - f(1)]/(e - 1) = (1 - 0)/(e - 1) = 1/(e - 1).$$

$$f(c) = 1/c = 1/(e - 1) \Rightarrow c = e - 1 \approx 1.718, \text{ which lies in } (1, e).$$

Answer: Verified; $c = e - 1$.

Problem 9. Find c of LMVT for $f(x) = \sqrt{x^2 - 4}$ on $[2, 3]$.

f is continuous on $[2, 3]$ and differentiable on $(2, 3)$.

$$[f(3) - f(2)]/(3 - 2) = \sqrt{5} - 0 = \sqrt{5}.$$

$$f'(x) = x/\sqrt{x^2 - 4} = \sqrt{5} \Rightarrow x^2 = 5(x^2 - 4) \Rightarrow 4x^2 = 20 \Rightarrow x = \sqrt{5} \approx 2.236.$$

Answer: $c = \sqrt{5}$, which lies in $(2, 3)$.

Problem 10. Using LMVT, prove that $(b - a)/(1 + b^2) < \tan^{-1}b - \tan^{-1}a < (b - a)/(1 + a^2)$ for $0 < a < b$. Hence show $\pi/4 + 3/25 < \tan^{-1}(4/3) < \pi/4 + 1/6$.

Apply LMVT to $f(x) = \tan^{-1}x$ on $[a, b]$: $\tan^{-1}b - \tan^{-1}a = (b - a)/(1 + c^2)$ for some $a < c < b$.

Since $a < c < b$, $1 + a^2 < 1 + c^2 < 1 + b^2$, so $1/(1 + b^2) < 1/(1 + c^2) < 1/(1 + a^2)$.

Multiply by $(b - a) > 0$.

$$\text{For } a = 1, b = 4/3: (1/3)/(1 + 16/9) < \tan^{-1}(4/3) - \pi/4 < (1/3)/2 \Rightarrow 3/25 < \tan^{-1}(4/3) - \pi/4 < 1/6.$$

Answer: Proved.

Problem 11. Using LMVT, show that $x/(1 + x) < \log(1 + x) < x$ for $x > 0$.

Apply LMVT to $f(t) = \log(1 + t)$ on $[0, x]$: $\log(1 + x) - 0 = x \cdot 1/(1 + c)$, $0 < c < x$.

Since $0 < c < x$, $1 < 1 + c < 1 + x \Rightarrow 1/(1 + x) < 1/(1 + c) < 1$.

Multiplying by $x > 0$: $x/(1 + x) < \log(1 + x) < x$.

Answer: Proved.

Problem 12. Verify Cauchy's MVT for $f(x) = e^x$ and $g(x) = e^{-x}$ on $[a, b]$.

Both are continuous and differentiable, and $g'(x) = -e^{-x} \neq 0$.

$$[f(b) - f(a)]/[g(b) - g(a)] = (e^b - e^a)/(e^{-b} - e^{-a}) = (e^b - e^a)/[(e^a - e^b)/e^{a+b}] = -e^{a+b}.$$

$$f(c)/g'(c) = e^c/(-e^{-c}) = -e^{2c}. \text{ Setting } -e^{2c} = -e^{a+b} \text{ gives } c = (a + b)/2.$$

Answer: $c = (a + b)/2$, the arithmetic mean of a and b .

Problem 13. Find c of Cauchy's MVT for $f(x) = \sqrt{x}$, $g(x) = 1/\sqrt{x}$ on $[a, b]$, $a > 0$.

$$[\sqrt{b} - \sqrt{a}]/[1/\sqrt{b} - 1/\sqrt{a}] = (\sqrt{b} - \sqrt{a})/[(\sqrt{a} - \sqrt{b})/\sqrt{ab}] = -\sqrt{ab}.$$

$$f(c)/g'(c) = (1/(2\sqrt{c})) / (-1/(2c^{3/2})) = -c.$$

Answer: $c = \sqrt{ab}$, the geometric mean.

Problem 14. Find the intervals where $f(x) = 2x^3 - 9x^2 + 12x + 5$ is increasing or decreasing.

$$f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2).$$

For $x < 1$: $f' > 0$. For $1 < x < 2$: $f' < 0$. For $x > 2$: $f' > 0$.

Answer: Increasing on $(-\infty, 1)$ and $(2, \infty)$; decreasing on $(1, 2)$.

Problem 15. Find the local maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 15$ using the first derivative test.

$$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3). \text{ Critical points: } x = 1, 3.$$

At $x = 1$: f' changes from $+$ ($x < 1$) to $-$ ($1 < x < 3$) $\Rightarrow$ local max. $f(1) = 1 - 6 + 9 + 15 = 19$.

At $x = 3$: f' changes from $-$ to $+$ $\Rightarrow$ local min. $f(3) = 27 - 54 + 27 + 15 = 15$.

Answer: Local maximum 19 at $x = 1$; local minimum 15 at $x = 3$.

Problem 16. Find the local extreme values of $f(x) = x^{2/3}(x - 5)$.

$$f(x) = x^{5/3} - 5x^{2/3}. f'(x) = (5/3)x^{2/3} - (10/3)x^{-1/3} = (5/3)x^{-1/3}(x - 2).$$

Critical points: $x = 2$ ($f' = 0$) and $x = 0$ (f' does not exist).

Sign of f' : $x < 0$: $x^{-1/3} < 0$ and $(x - 2) < 0 \Rightarrow f' > 0$. $0 < x < 2$: $f' < 0$. $x > 2$: $f' > 0$.

At $x = 0$: $+$ to $- \Rightarrow$ local max, $f(0) = 0$. At $x = 2$: $-$ to $+$ $\Rightarrow$ local min, $f(2) = 2^{2/3}(-3) = -3 \cdot \sqrt[3]{4} \approx -4.76$.

Answer: Local max 0 at $x = 0$ (a cusp); local min $-3\sqrt[3]{4}$ at $x = 2$.

Problem 17. Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on $[-1/2, 4]$.

$$f'(x) = 3x^2 - 6x = 3x(x - 2). \text{ Critical points in the interval: } x = 0, 2.$$

$$f(-1/2) = -1/8 - 3/4 + 1 = 1/8; f(0) = 1; f(2) = 8 - 12 + 1 = -3; f(4) = 64 - 48 + 1 = 17.$$

Answer: Absolute maximum 17 at $x = 4$; absolute minimum -3 at $x = 2$.

Problem 18. Show that of all rectangles with a given perimeter $2p$, the square has maximum area.

Let the sides be x and $p - x$. Area $A(x) = x(p - x) = px - x^2$, $0 < x < p$.

$$A'(x) = p - 2x = 0 \Rightarrow x = p/2. A' \text{ changes from } + \text{ to } - \text{ at } p/2 \Rightarrow \text{maximum.}$$

Then the other side $= p - p/2 = p/2$ as well.

Answer: Maximum area occurs when both sides are $p/2$, i.e. the rectangle is a square; $A_{\max} = p^2/4$.

Problem 19. Find the radius of convergence and interval of convergence of $\sum_{n=1}^{\infty} x^n/(n \cdot 3^n)$.

$$c_n = 1/(n \cdot 3^n). R = \lim |c_n/c_{n+1}| = \lim [(n+1)3^{n+1}]/(n \cdot 3^n) = 3.$$

At $x = 3$: $\sum 1/n$ diverges. At $x = -3$: $\sum (-1)^n/n$ converges (alternating series).

Answer: $R = 3$; interval of convergence $[-3, 3)$.

Problem 20. Find the radius of convergence of $\sum n! x^n/n^n$.

$$|c_n/c_{n+1}| = [n!/n^n] \times [(n+1)^{n+1}/(n+1)!] = (n+1)^n/n^n = (1 + 1/n)^n \rightarrow e.$$

Answer: $R = e$.

Problem 21. Expand e^x in Maclaurin's series.

$$f^{(n)}(x) = e^x \text{ for all } n, \text{ so } f^{(n)}(0) = 1.$$

$$f(x) = f(0) + x f'(0) + x^2/2! f''(0) + \dots = 1 + x + x^2/2! + x^3/3! + \dots$$

Answer: $e^x = \sum x^n/n!$, valid for all real x .

Problem 22. Expand $\log(1+x)$ in Maclaurin's series up to the term in x^4 .

$$f(x) = \log(1+x), f(0) = 0. f' = 1/(1+x), f'(0) = 1. f'' = -1/(1+x)^2, f''(0) = -1.$$

$$f''' = 2/(1+x)^3, f'''(0) = 2. f^{(4)} = -6/(1+x)^4, f^{(4)}(0) = -6.$$

$$\log(1+x) = 0 + x - x^2/2! + 2x^3/3! - 6x^4/4! = x - x^2/2 + x^3/3 - x^4/4.$$

Answer: $\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$

Problem 23. Expand $\tan x$ in Maclaurin's series up to x^5 .

$$y = \tan x, y(0) = 0. y' = \sec^2 x = 1 + y^2, y'(0) = 1.$$

$$y'' = 2yy', y''(0) = 0. y''' = 2y'^2 + 2yy'', y'''(0) = 2.$$

$$y^{(4)} = 6y'y'' + 2yy''', y^{(4)}(0) = 0. y^{(5)} = 6y''^2 + 8y'y''' + 2yy^{(4)}, y^{(5)}(0) = 0 + 16 + 0 = 16.$$

$$\tan x = x + 2x^3/3! + 16x^5/5! = x + x^3/3 + 2x^5/15.$$

Answer: $\tan x = x + x^3/3 + 2x^5/15 + \dots$

Problem 24. Expand $e^{\sin x}$ by Maclaurin's series up to the term in x^4 .

$$y = e^{\sin x}: y(0) = 1. y' = y \cos x: y'(0) = 1.$$

$$y'' = y' \cos x - y \sin x: y''(0) = 1. y''' = y'' \cos x - 2y' \sin x - y \cos x: y'''(0) = 1 - 0 - 1 = 0.$$

$$y^{(4)} = y''' \cos x - 3y'' \sin x - 3y' \cos x + y \sin x: y^{(4)}(0) = 0 - 0 - 3 + 0 = -3.$$

$$e^{\sin x} = 1 + x + x^2/2 + 0 \cdot x^3 - 3x^4/24.$$

Answer: $e^{\sin x} = 1 + x + x^2/2 - x^4/8 + \dots$

Problem 25. Expand $\sin x$ in powers of $(x - \pi/2)$ (Taylor series about $\pi/2$).

$$f(\pi/2) = 1, f'(\pi/2) = \cos(\pi/2) = 0, f''(\pi/2) = -\sin(\pi/2) = -1, f'''(\pi/2) = 0, f^{(4)}(\pi/2) = 1.$$

Answer: $\sin x = 1 - (x - \pi/2)^2/2! + (x - \pi/2)^4/4! - \dots$

Problem 26. Expand $\log x$ in powers of $(x - 1)$ and hence find $\log(1.1)$ to four decimal places.

$$f(1) = 0, f' = 1/x \Rightarrow 1, f'' = -1/x^2 \Rightarrow -1, f''' = 2/x^3 \Rightarrow 2, f^{(4)} = -6/x^4 \Rightarrow -6.$$

$$\log x = (x - 1) - (x - 1)^2/2 + (x - 1)^3/3 - (x - 1)^4/4 + \dots$$

$$\text{For } x = 1.1: 0.1 - 0.005 + 0.000333 - 0.000025 = 0.095308.$$



Answer: $\log(1.1) \approx 0.0953$.

Problem 27. Using Taylor's series, find $\sqrt{10}$ correct to three decimal places (expand $\sqrt{x}$ about $x = 9$).

$$f(9) = 3; f'(x) = 1/(2\sqrt{x}) \Rightarrow 1/6; f''(x) = -1/(4x^{3/2}) \Rightarrow -1/108; f''' = 3/(8x^{5/2}) \Rightarrow 3/1944 = 1/648.$$

$$\sqrt{10} = 3 + 1/6 - (1/108)(1/2) + (1/648)(1/6) = 3 + 0.166667 - 0.004630 + 0.000257.$$

Answer: $\sqrt{10} \approx 3.1623$ (true value 3.16228).

Problem 28. Evaluate $\lim_{x \rightarrow 0} (e^x - 1 - x)/x^2$ using series.

$$e^x - 1 - x = x^2/2 + x^3/6 + \dots$$

Answer: Limit = $1/2$.

Problem 29. Write Taylor's formula with Lagrange remainder for $f(x) = e^x$ about 0 with $n = 3$ and estimate the error when $e^{0.1}$ is approximated by $1 + 0.1 + 0.01/2 + 0.001/6$.

$$R_3 = (x^4/4!) e^{\theta x}, 0 < \theta < 1. \text{ For } x = 0.1: |R_3| \leq (0.0001/24) e^{0.1} \approx 0.0000046.$$

Answer: The approximation 1.105167 is correct to about 5 decimal places (error $< 5 \times 10^{-6}$).

2.17 Practice problems with solutions: Module II

Q1. Verify Rolle's theorem for $f(x) = x^2$ on $[-1, 1]$.

Continuous and differentiable; $f(-1) = f(1) = 1$. $f'(c) = 2c = 0 \Rightarrow c = 0 \in (-1, 1)$.

Answer: $c = 0$.

Q2. Verify Rolle's theorem for $f(x) = \sin x$ on $[0, \pi]$.

$$f(0) = f(\pi) = 0; f'(c) = \cos c = 0 \Rightarrow c = \pi/2 \in (0, \pi).$$

Answer: $c = \pi/2$.

Q3. Verify Rolle's theorem for $f(x) = (x - a)^m(x - b)^n$ on $[a, b]$, m, n positive integers.

$$\text{Polynomial; } f(a) = f(b) = 0. f'(x) = (x - a)^{m-1}(x - b)^{n-1}[m(x - b) + n(x - a)].$$

$$f'(c) = 0 \text{ with } c \neq a, b \Rightarrow m(c - b) + n(c - a) = 0 \Rightarrow c = (mb + na)/(m + n), \text{ which divides } [a, b] \text{ internally in the ratio } m : n.$$

Answer: $c = (mb + na)/(m + n)$.

Q4. Is Rolle's theorem applicable to $f(x) = 1 - (x - 1)^{2/3}$ on $[0, 2]$?

$$f(0) = 1 - 1 = 0, f(2) = 0. \text{ But } f'(x) = -(2/3)(x - 1)^{-1/3} \text{ does not exist at } x = 1 \in (0, 2).$$

Answer: Not applicable.

Q5. Find c of LMVT for $f(x) = x^2$ on $[1, 3]$.

$$(9 - 1)/2 = 4 = 2c \Rightarrow c = 2.$$



Answer: $c = 2$.

Q6. Find c of LMVT for $f(x) = e^x$ on $[0, 1]$.

$$(e - 1)/1 = e^c \Rightarrow c = \log(e - 1) \approx 0.541 \in (0, 1).$$

Answer: $c = \log(e - 1)$.

Q7. Prove $|\sin a - \sin b| \leq |a - b|$ for all real a, b .

By LMVT on $\sin x$: $\sin a - \sin b = (a - b) \cos c$ for some c between a and b .

$$\text{Since } |\cos c| \leq 1, |\sin a - \sin b| = |a - b| |\cos c| \leq |a - b|.$$

Answer: Proved.

Q8. Find c of Cauchy's MVT for $f = 1/x^2$, $g = 1/x$ on $[a, b]$, $0 < a < b$.

$$[1/b^2 - 1/a^2]/[1/b - 1/a] = (1/b + 1/a) = (a + b)/(ab).$$

$$f(c)/g'(c) = (-2/c^3)/(-1/c^2) = 2/c. \text{ So } 2/c = (a + b)/(ab).$$

Answer: $c = 2ab/(a + b)$ (harmonic mean).

Q9. Find the local maxima and minima of $f(x) = x^4 - 8x^2 + 3$.

$$f' = 4x^3 - 16x = 4x(x - 2)(x + 2). \text{ Critical points } 0, \pm 2.$$

Sign of f' : $(-\infty, -2) -$; $(-2, 0) +$; $(0, 2) -$; $(2, \infty) +$.

$$x = -2: \text{min, } f = 16 - 32 + 3 = -13. \quad x = 0: \text{max, } f = 3. \quad x = 2: \text{min, } f = -13.$$

Answer: Local max 3 at $x = 0$; local min -13 at $x = \pm 2$.

Q10. Find the absolute max and min of $f(x) = \sin x + \cos x$ on $[0, \pi]$.

$$f' = \cos x - \sin x = 0 \Rightarrow x = \pi/4. \quad f(0) = 1, \quad f(\pi/4) = \sqrt{2}, \quad f(\pi) = -1.$$

Answer: Absolute max $\sqrt{2}$ at $\pi/4$; absolute min -1 at π .

Q11. Find the radius and interval of convergence of $\sum (x - 2)^n/n^2$.

$R = \lim (n + 1)^2/n^2 = 1$. At $x = 3$: $\sum 1/n^2$ converges. At $x = 1$: $\sum (-1)^n/n^2$ converges absolutely.

Answer: $R = 1$; interval $[1, 3]$.

Q12. Find the Maclaurin series of $\cos x$ up to x^6 .

$$f^{(n)}(0) \text{ cycles } 1, 0, -1, 0, 1, 0, -1.$$

Answer: $\cos x = 1 - x^2/2 + x^4/24 - x^6/720 + \dots$

Q13. Expand $e^x \cos x$ up to x^4 .

$$(1 + x + x^2/2 + x^3/6 + x^4/24)(1 - x^2/2 + x^4/24).$$

$$x^2: 1/2 - 1/2 = 0. \quad x^3: 1/6 - 1/2 = -1/3. \quad x^4: 1/24 - 1/4 + 1/24 = -1/6.$$

Answer: $e^x \cos x = 1 + x - x^3/3 - x^4/6 + \dots$

Q14. Expand $\log(1 + \sin x)$ up to x^4 .



Let $s = \sin x = x - x^3/6$. $\log(1 + s) = s - s^2/2 + s^3/3 - s^4/4$.

$s^2 = x^2 - x^4/3$, $s^3 = x^3$, $s^4 = x^4$ (to order 4).

$= x - x^3/6 - x^2/2 + x^4/6 + x^3/3 - x^4/4 = x - x^2/2 + x^3/6 - x^4/12$.

Answer: $\log(1 + \sin x) = x - x^2/2 + x^3/6 - x^4/12 + \dots$

Q15. Expand $\sin^{-1}x$ up to x^5 .

$d/dx \sin^{-1}x = (1 - x^2)^{-1/2} = 1 + x^2/2 + 3x^4/8 + \dots$

Integrate term by term with $\sin^{-1}0 = 0$.

Answer: $\sin^{-1}x = x + x^3/6 + 3x^5/40 + \dots$

Q16. Expand e^x in powers of $(x - 1)$.

All derivatives at $x = 1$ equal e .

Answer: $e^x = e[1 + (x - 1) + (x - 1)^2/2! + (x - 1)^3/3! + \dots]$.

Q17. Evaluate $\lim_{x \rightarrow 0} (x - \sin x)/(x - \tan x)$.

$x - \sin x = x^3/6 - \dots$; $x - \tan x = -x^3/3 - \dots$

Answer: $(1/6)/(-1/3) = -1/2$.

2.18 MCQs: Module II

1. Which is NOT a condition of Rolle's theorem?

(a) $f(a) = f(b)$ (b) f continuous on $[a, b]$ (c) $f(a) = f(b)$ (d) f differentiable on (a, b)

☐ (a) — The three conditions are continuity, differentiability and $f(a) = f(b)$.

2. Rolle's theorem guarantees a point c where

(a) $f'(c) = 0$ (b) $f(c) = 0$ (c) $f(c) = f(a) + f(b)$ (d) $f''(c) = 0$

☐ (a) — Horizontal tangent.

3. For $f(x) = x^2 - 4x + 3$ on $[1, 3]$, the Rolle's point c is

(a) 2.5 (b) 2 (c) 3 (d) 1.5

☐ (b) — $f'(c) = 2c - 4 = 0$.

4. Rolle's theorem is not applicable to $f(x) = |x|$ on $[-1, 1]$ because

(a) f is not differentiable at 0 (b) f is not bounded (c) f is not continuous (d) $f(-1) \neq f(1)$

☐ (a) — Corner at $x = 0$.

5. Rolle's theorem is not applicable to $\tan x$ on $[0, \pi]$ because

(a) $\tan x$ is discontinuous at $\pi/2$ (b) it is periodic (c) $\tan 0 \neq \tan \pi$ (d) $\tan x$ is not differentiable at 0

☐ (a) — Infinite discontinuity.

6. The geometrical meaning of LMVT is that the tangent at some point is

(a) parallel to the chord joining end points (b) perpendicular to the chord (c) parallel to the y -axis (d) parallel to the x -axis

☐ (a) — Slope $f'(c) = \text{slope of chord}$.

7. LMVT becomes Rolle's theorem when

(a) $f(a) = f(b)$ (b) f is linear (c) $a = b$ (d) $f'(a) = 0$

☐ (a) — Then slope of chord = 0.

8. The value of c in LMVT for $f(x) = x^2$ on $[2, 4]$ is

(a) 4 (b) 3 (c) 3.5 (d) 2.5

☐ (b) — For a quadratic, $c = \text{midpoint}$.



- 9.** The value of c in LMVT for $f(x) = \log x$ on $[1, e]$ is
(a) e (b) $1/e$ (c) $e - 1$ (d) $(e + 1)/2$
☐ (c) — $1/c = 1/(e - 1)$.
- 10.** The value of c in LMVT for $f(x) = x^3$ on $[0, 3]$ is
(a) 1 (b) $3/2$ (c) 2 (d) $\sqrt{3}$
☐ (d) — $3c^2 = 27/3 = 9$.
- 11.** If $f(x) = 0$ for all x in (a, b) , then f is
(a) decreasing (b) zero (c) increasing (d) constant
☐ (d) — Consequence of LMVT.
- 12.** If $f(x) > 0$ on (a, b) , then f is
(a) strictly decreasing (b) periodic (c) constant (d) strictly increasing
☐ (d) — Consequence of LMVT.
- 13.** In the form $f(a + h) = f(a) + h f'(a + \theta h)$, θ satisfies
(a) $\theta = 1/2$ always (b) $0 < \theta < 1$ (c) $\theta > 1$ (d) $\theta < 0$
☐ (b) — $c = a + \theta h$ lies strictly between a and $a + h$.
- 14.** For $f(x) = x^2$ in LMVT form $f(a + h) = f(a) + hf'(a + \theta h)$, θ equals
(a) 1 (b) $1/4$ (c) $1/2$ (d) $1/3$
☐ (c) — $(a + h)^2 - a^2 = 2h(a + \theta h)$.
- 15.** In Cauchy's MVT with $f = e^x$, $g = e^{-x}$, c equals
(a) $(a + b)/2$ (b) $a + b$ (c) $\sqrt{ab}$ (d) $2ab/(a + b)$
☐ (a) — Arithmetic mean.
- 16.** In Cauchy's MVT with $f = \sqrt{x}$, $g = 1/\sqrt{x}$, c equals
(a) ab (b) $\sqrt{ab}$ (c) $2ab/(a + b)$ (d) $(a + b)/2$
☐ (b) — Geometric mean.
- 17.** In Cauchy's MVT with $f = 1/x^2$, $g = 1/x$, c equals
(a) $\sqrt{ab}$ (b) $(a + b)/2$ (c) $2ab/(a + b)$ (d) a/b
☐ (c) — Harmonic mean.
- 18.** Cauchy's MVT reduces to LMVT when $g(x) =$
(a) x^2 (b) 1 (c) x (d) e^x
☐ (c) — $g'(x) = 1$.
- 19.** A critical point of f is a point where
(a) $f(c) = 0$ (b) $f'(c) = 0$ or $f'(c)$ does not exist (c) $f''(c) = 0$ (d) f is discontinuous
☐ (b) — Definition.
- 20.** If f changes sign from $+$ to $-$ at c , then f has
(a) no extremum (b) local minimum (c) local maximum (d) point of inflection
☐ (c) — First derivative test.
- 21.** $f(x) = x^3$ at $x = 0$ has
(a) local minimum (b) neither (inflection point) (c) local maximum (d) absolute maximum
☐ (b) — f' does not change sign.
- 22.** The function $f(x) = x^3 - 3x$ has a local maximum at
(a) $x = 3$ (b) $x = -1$ (c) $x = 0$ (d) $x = 1$
☐ (b) — $f'(-1) = -6 < 0$.
- 23.** The local minimum value of $f(x) = x^2 - 4x + 7$ is
(a) 2 (b) 4 (c) 3 (d) 7
☐ (c) — At $x = 2$, $f = 4 - 8 + 7 = 3$.



- 24.** The absolute maximum of $f(x) = x^2$ on $[-2, 1]$ is
(a) 4 (b) 0 (c) 1 (d) 2
□ (a) — At the end point $x = -2$.
- 25.** $f(x) = e^x$ is
(a) decreasing everywhere (b) increasing only for $x > 0$ (c) increasing everywhere (d) constant
□ (c) — $f' = e^x > 0$.
- 26.** The series $\sum 1/n$ is
(a) convergent to 1 (b) oscillatory (c) convergent (d) divergent
□ (d) — Harmonic series ($p = 1$).
- 27.** The geometric series $\sum r^n$ converges if
(a) $|r| > 1$ (b) $r = 1$ (c) $|r| < 1$ (d) $r = -1$
□ (c) — Sum = $1/(1 - r)$.
- 28.** The radius of convergence of $\sum x^n$ is
(a) 1 (b) ∞ (c) $1/2$ (d) 0
□ (a) — Geometric series.
- 29.** The radius of convergence of $\sum x^n/n!$ is
(a) e (b) ∞ (c) 1 (d) 0
□ (b) — Ratio $\rightarrow 0$.
- 30.** The radius of convergence of $\sum n! x^n$ is
(a) ∞ (b) 1 (c) e (d) 0
□ (d) — Converges only at $x = 0$.
- 31.** The radius of convergence of $\sum x^n/2^n$ is
(a) 1 (b) $1/2$ (c) 4 (d) 2
□ (d) — $|c_n/c_{n+1}| = 2$.
- 32.** The interval of convergence of $\sum x^n/n$ is
(a) $[-1, 1)$ (b) $(-1, 1)$ (c) $[-1, 1]$ (d) $(-1, 1]$
□ (a) — Converges at -1 , diverges at 1 .
- 33.** The ratio test fails when the limit L equals
(a) 2 (b) 0 (c) ∞ (d) 1
□ (d) — $L = 1$ is inconclusive.
- 34.** The Maclaurin series is the Taylor series about
(a) $x = a$ (b) $x = 1$ (c) $x = \infty$ (d) $x = 0$
□ (d) — $a = 0$.
- 35.** The coefficient of x^3 in the Maclaurin series of e^x is
(a) 1 (b) $1/2$ (c) $1/6$ (d) $1/3$
□ (c) — $1/3!$.
- 36.** The coefficient of x^3 in $\sin x$ is
(a) $-1/3$ (b) $1/3$ (c) $1/6$ (d) $-1/6$
□ (d) — $-x^3/3!$.
- 37.** The coefficient of x^4 in $\cos x$ is
(a) $1/4$ (b) $1/24$ (c) $-1/24$ (d) $1/12$
□ (b) — $+x^4/4!$.
- 38.** The coefficient of x^3 in $\tan x$ is
(a) $1/3$ (b) $2/15$ (c) 1 (d) $1/6$
□ (a) — $\tan x = x + x^3/3 + \dots$



- 39.** $\log(1 + x) = x - x^2/2 + x^3/3 - \dots$ is valid for
(a) $x > 0$ only (b) $-1 < x \leq 1$ (c) $|x| > 1$ (d) all x
□ (b) — Interval of convergence.
- 40.** The coefficient of x^2 in $(1 + x)^{1/2}$ is
(a) $1/2$ (b) $1/8$ (c) $-1/8$ (d) $-1/2$
□ (c) — $n(n - 1)/2 = (1/2)(-1/2)/2$.
- 41.** The Maclaurin series of $1/(1 - x)$ is
(a) $1 + x + x^2 + \dots$ (b) $x + x^2 + \dots$ (c) $1 + x/1! + x^2/2! + \dots$ (d) $1 - x + x^2 - \dots$
□ (a) — Geometric series.
- 42.** $\lim_{x \rightarrow 0} (\sin x - x)/x^3$ equals
(a) 0 (b) $-1/6$ (c) 1 (d) $1/6$
□ (b) — $\sin x - x = -x^3/6 + \dots$
- 43.** $\lim_{x \rightarrow 0} (1 - \cos x)/x^2$ equals
(a) 2 (b) $1/2$ (c) 1 (d) 0
□ (b) — $1 - \cos x = x^2/2 - \dots$
- 44.** The Lagrange form of the remainder after n terms in Taylor's theorem is
(a) $f^{(n+1)}(a)$ (b) $h^{n+1}f^{(n+1)}(a + \theta h)/(n + 1)!$ (c) $h f(a + \theta h)$ (d) $h^n f^{(n)}(a)/n!$
□ (b) — Standard form.
- 45.** Taylor's theorem with $n = 0$ reduces to
(a) Fermat's theorem (b) Rolle's theorem (c) LMVT (d) CMVT
□ (c) — $f(a + h) = f(a) + hf'(a + \theta h)$.
- 46.** The first two non-zero terms of the Maclaurin series of $\sinh x$ are
(a) $1 + x^2/2$ (b) $x + x^3/6$ (c) $x + x^2/2$ (d) $x - x^3/6$
□ (b) — $\sinh x = x + x^3/3! + \dots$
- 47.** Using series, $e^{0.1}$ is approximately
(a) 1.2 (b) 1.01 (c) 1.1 (d) 1.105
□ (d) — $1 + 0.1 + 0.005 + 0.000167$.
- 48.** The Taylor expansion of $f(x)$ about $x = a$ involves
(a) f and its derivatives at 0 (b) f and its derivatives at a (c) only $f(a)$ (d) only $f'(a)$
□ (b) — Coefficients are $f^{(n)}(a)/n!$.
- 49.** The equation $x^3 + x - 1 = 0$ has how many real roots?
(a) 1 (b) 2 (c) 0 (d) 3
□ (a) — $f'(x) = 3x^2 + 1 > 0$ so f is strictly increasing.

Module III: Multivariable Calculus (Differentiation)

In engineering, quantities usually depend on more than one variable. The temperature of a plate depends on x and y ; the volume of a gas depends on pressure and temperature. This module extends differentiation to such functions and uses it to find maxima, minima and constrained optima.

3.1 Functions of several variables, limits and continuity

A function of two variables $z = f(x, y)$ assigns a real number to each point (x, y) of a region D in the plane. Its graph is a surface in 3-D space. A **level curve** $f(x, y) = k$ joins points where f has the same value (like contour lines on a map).

Limit. $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ if $f(x, y)$ gets arbitrarily close to L as (x, y) approaches (a, b) along **every** path.

Two-path test. If f approaches different values along two different paths, the limit does not exist. Example: $f(x, y) = xy/(x^2 + y^2)$ at $(0, 0)$. Along $y = mx$: $f = m/(1 + m^2)$, which depends on m . So the limit does not exist.

Continuity. f is continuous at (a, b) if $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$.

3.2 Partial derivatives

The partial derivative of f with respect to x is found by treating y as a constant:

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \rightarrow 0} [f(x + h, y) - f(x, y)]/h \quad \frac{\partial f}{\partial y} = f_y = \lim_{k \rightarrow 0} [f(x, y + k) - f(x, y)]/k$$

Geometrical meaning. $f_x(a, b)$ is the slope of the curve cut from the surface $z = f(x, y)$ by the plane $y = b$.

Higher-order partial derivatives.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

Clairaut's (Schwarz's) theorem. If f_{xy} and f_{yx} are continuous near (a, b) , then $f_{xy}(a, b) = f_{yx}(a, b)$. For all functions in this course, mixed partials are equal.

Homogeneous functions and Euler's theorem. $f(x, y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$. Then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f \quad x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n - 1) f$$

Example: $f = x^3 + y^3 + 3x^2y$ is homogeneous of degree 3, so $x f_x + y f_y = 3f$.

Laplace's equation. A function $u(x, y)$ satisfying $u_{xx} + u_{yy} = 0$ is called **harmonic**.

Example: $u = \log(x^2 + y^2)$, $u = e^x \cos y$, $u = x^2 - y^2$.

3.3 Total derivative and chain rule

Total differential: $dz = (\partial z / \partial x) dx + (\partial z / \partial y) dy$

If $z = f(x, y)$, $x = x(t)$, $y = y(t)$: $dz/dt = (\partial z / \partial x)(dx/dt) + (\partial z / \partial y)(dy/dt)$

If $z = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$: $\partial z / \partial u = z_x x_u + z_y y_u$, $\partial z / \partial v = z_x x_v + z_y y_v$

Implicit function $f(x, y) = 0$: $dy/dx = -f_x / f_y$

Approximation and errors. Since $\Delta z \approx dz$, small errors in x and y produce an error in z of about $z_x \Delta x + z_y \Delta y$. Percentage error in $z = 100 \times dz/z$.

3.4 Jacobians

If $u = u(x, y)$ and $v = v(x, y)$, the **Jacobian** of u, v with respect to x, y is the determinant

$$J = \partial(u, v)/\partial(x, y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

For three functions u, v, w of x, y, z , $J = \partial(u, v, w)/\partial(x, y, z)$ is the 3×3 determinant of all first partial derivatives (rows: u, v, w ; columns: x, y, z).

Meaning. The Jacobian measures how a small area (or volume) changes under the transformation: $du dv = |J| dx dy$. It is used when changing variables in double and triple integrals.

Properties.

- (1) $J J' = 1$, where $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$
- (2) Chain rule: $\partial(u, v)/\partial(x, y) = \partial(u, v)/\partial(r, s) \times \partial(r, s)/\partial(x, y)$
- (3) u, v are **functionally dependent** (related by some $F(u, v) = 0$) if and only if $\partial(u, v)/\partial(x, y) = 0$ identically

Standard Jacobians (memorise):

Polar: $x = r \cos \theta, y = r \sin \theta \Rightarrow \partial(x, y)/\partial(r, \theta) = r ; \partial(r, \theta)/\partial(x, y) = 1/r$

Cylindrical: $x = r \cos \theta, y = r \sin \theta, z = z \Rightarrow \partial(x, y, z)/\partial(r, \theta, z) = r$

Spherical: $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta \Rightarrow \partial(x, y, z)/\partial(r, \theta, \phi) = r^2 \sin \theta$

3.5 Hessian matrix

The Hessian matrix of $f(x, y)$ is the matrix of second partial derivatives:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \det H = f_{xx} f_{yy} - (f_{xy})^2$$

For $f(x, y, z)$, H is 3×3 : rows $(f_{xx}, f_{xy}, f_{xz}), (f_{yx}, f_{yy}, f_{yz}), (f_{zx}, f_{zy}, f_{zz})$. Because mixed partials are equal, H is always **symmetric**.

The Hessian plays the role of the second derivative for functions of several variables. At a critical point:

- H positive definite (all leading principal minors > 0) $\Rightarrow$ local minimum.
- H negative definite (leading minors alternate in sign, starting negative: $D_1 < 0, D_2 > 0, D_3 < 0$) $\Rightarrow$ local maximum.
- H indefinite ($\det H < 0$ for two variables) $\Rightarrow$ saddle point.
- $\det H = 0 \Rightarrow$ test is inconclusive.

3.6 Maxima, minima and saddle points (two variables)

Definitions. f has a local maximum at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) near (a, b) ; a local minimum if $f(a, b) \leq f(x, y)$ near (a, b) . A **saddle point** is a critical point where f has neither: in some directions f increases and in others it decreases (like a horse saddle or a mountain pass). The extreme values are called **extrema**; the points are **stationary (critical) points**.

Necessary condition. If f has a local extremum at (a, b) and the first partials exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Second derivative test (Lagrange's conditions). At a stationary point (a, b) , let

$r = f_{xx}(a, b), s = f_{xy}(a, b), t = f_{yy}(a, b), D = rt - s^2 (= \det H)$
 $D > 0$ and $r < 0 \Rightarrow$ local MAXIMUM $D > 0$ and $r > 0 \Rightarrow$ local MINIMUM
 $D < 0 \Rightarrow$ SADDLE POINT (no extremum) $D = 0 \Rightarrow$ test FAILS (investigate further)

Working rule.

- Find f_x and f_y ; solve $f_x = 0$ and $f_y = 0$ simultaneously to get all stationary points.
- Find r, s, t and $D = rt - s^2$ at each stationary point.
- Classify each point using the table above and find the value of f .

Three variables. For $f(x, y, z)$, solve $f_x = f_y = f_z = 0$. Form the 3×3 Hessian and check the leading minors $D_1 = f_{xx}, D_2 = 2 \times 2$ determinant, $D_3 = \det H$.

3.7 Method of Lagrange multipliers (constrained extrema)

Often we must maximise or minimise $f(x, y, z)$ subject to a condition (constraint) $\varphi(x, y, z) = 0$. Example: maximise the volume of a box with a fixed surface area.

Idea. At a constrained extremum, the level surface of f touches the constraint surface, so their normals are parallel: $\nabla f = \lambda \nabla \varphi$. The number λ is called the **Lagrange multiplier**.

Method.

Form the Lagrangian function $F(x, y, z, \lambda) = f(x, y, z) + \lambda \varphi(x, y, z)$
 Solve: $F_x = f_x + \lambda \varphi_x = 0, F_y = f_y + \lambda \varphi_y = 0, F_z = f_z + \lambda \varphi_z = 0, \varphi(x, y, z) = 0$

- Four equations in four unknowns x, y, z, λ . Usually eliminate λ first by dividing or comparing the equations.
- The solutions are the stationary points. Decide max or min by physical reasoning (or by comparing values).
- For two constraints $\varphi = 0$ and $\psi = 0$, use $F = f + \lambda \varphi + \mu \psi$.

Advantages: constraint need not be solved explicitly; symmetric problems become easy.

Limitation: the method does not directly tell whether the point is a max or a min.

Note: In exam answers, always write the Lagrangian, the three (or four) equations, the elimination steps and a clear final statement such as "the maximum volume is $a^3/(3\sqrt{3})$ at $x = y = z = a/\sqrt{3}$ ".

3.8 Geometric meaning of partial derivatives and the tangent plane

The surface $z = f(x, y)$ is cut by the vertical plane $y = b$ in a curve. The slope of that curve at $x = a$ is $f_x(a, b)$. Similarly $f_y(a, b)$ is the slope of the curve cut by the plane $x = a$. These two slopes determine the **tangent plane** to the surface at $P(a, b, f(a, b))$:

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The tangent plane is the best linear approximation of the surface near P . This is the two-variable version of the tangent line, and it leads directly to the total differential $dz = f_x dx + f_y dy$.

Gradient. The vector $\nabla f = (f_x, f_y)$ points in the direction in which f increases fastest, and its magnitude is that maximum rate. It is perpendicular to the level curve $f(x, y) = c$ through the point. (The gradient reappears in Physics Module III, Electromagnetism.)

Directional derivative. The rate of change of f in the direction of a unit vector u is $D_u f = \nabla f \cdot u$. It is largest when u points along ∇f .

3.9 Homogeneous functions: proof of Euler's theorem

Let f be homogeneous of degree n , so $f(tx, ty) = t^n f(x, y)$ for all $t > 0$. Differentiate both sides with respect to t using the chain rule: $x f_x(tx, ty) + y f_y(tx, ty) = n t^{n-1} f(x, y)$. Put $t = 1$: $x f_x + y f_y = n f$.

Corollary. Differentiating Euler's relation again with respect to x and y and combining gives $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f$.

Extension. If $z = F(u)$ where u is homogeneous of degree n , then $x z_x + y z_y = n F(u)/F'(u)$. This is how problems like $u = \sin^{-1}[(x^2 + y^2)/(x + y)]$ are solved.

3.10 Why the second derivative test works (Taylor's formula in two variables)

Near a stationary point (a, b) , $f_x = f_y = 0$, so Taylor's formula in two variables gives

$$f(a+h, b+k) - f(a, b) \approx (1/2)[r h^2 + 2s hk + t k^2], \text{ with } r = f_{xx}, s = f_{xy}, t = f_{yy} \text{ at } (a, b)$$

The sign of the change depends on the quadratic form $Q = rh^2 + 2shk + tk^2$. Completing the square (when $r \neq 0$): $Q = (1/r)[(rh + sk)^2 + (rt - s^2)k^2]$.

- If $rt - s^2 > 0$ and $r > 0$, $Q > 0$ for all small $(h, k) \neq 0 \Rightarrow f$ increases in every direction $\Rightarrow$ minimum.
- If $rt - s^2 > 0$ and $r < 0$, $Q < 0 \Rightarrow$ maximum.
- If $rt - s^2 < 0$, Q takes both signs $\Rightarrow$ saddle point.
- If $rt - s^2 = 0$, higher-order terms decide.

This is exactly the statement that the Hessian matrix H is positive definite, negative definite or indefinite.

3.11 Hessian test for three variables

For $f(x, y, z)$, at a stationary point form the Hessian $H = [f_{xx} f_{xy} f_{xz}; f_{yx} f_{yy} f_{yz}; f_{zx} f_{zy} f_{zz}]$ and its leading principal minors $D_1 = f_{xx}$, $D_2 = f_{xx} f_{yy} - f_{xy}^2$, $D_3 = |H|$.

Signs of D_1, D_2, D_3	Nature
+, +, +	Local minimum (positive definite)
-, +, -	Local maximum (negative definite)
Any other pattern with $D_3 \neq 0$	Saddle point
$D_3 = 0$	Test inconclusive

Example. $f = x^2 + y^2 + z^2 - xy + x - 2z$. $f_x = 2x - y + 1 = 0$, $f_y = 2y - x = 0$, $f_z = 2z - 2 = 0 \Rightarrow (-2/3, -1/3, 1)$. $H = [2 \ -1 \ 0; -1 \ 2 \ 0; 0 \ 0 \ 2]$. $D_1 = 2$, $D_2 = 3$, $D_3 = 6$, all positive $\Rightarrow$ minimum. $f_{\min} = 4/9 + 1/9 + 1 - 2/9 - 2/3 - 2 = -4/3$.

3.12 Geometrical interpretation of Lagrange multipliers

Suppose we want to maximise $f(x, y)$ on the curve $\phi(x, y) = 0$. Draw the level curves $f = c$ for increasing values of c . The largest c whose level curve still meets the constraint curve is the maximum. At that point the level curve just **touches** the constraint curve, so the two curves have a common tangent and therefore parallel normals. Normals are given by gradients, so $\nabla f = -\lambda \nabla \phi$ for some λ . Writing this component-wise together with $\phi = 0$ gives exactly the Lagrange equations.

Meaning of λ . λ measures how much the optimum value of f changes when the constraint is relaxed slightly. In economics it is called the shadow price. In mechanics it is a force of constraint.

3.13 Engineering applications of multivariable calculus

- **Error analysis:** $dz = z_x dx + z_y dy$ estimates the effect of measurement errors (see the pendulum problem).
- **Thermodynamics:** partial derivatives like $(\partial P/\partial T)_V$ describe how pressure changes with temperature at constant volume.
- **Optimisation in design:** minimum material for a given capacity, maximum power transfer, least-cost design problems use maxima/minima and Lagrange multipliers.
- **Change of variables:** Jacobians convert integrals to polar, cylindrical and spherical coordinates (areas of circles, volumes of spheres, fields around wires).
- **Machine learning:** gradient descent moves opposite to ∇f to minimise an error function; the Hessian determines the curvature of the error surface.

3.14 Common mistakes to avoid in Module III

- Missing stationary points. Solve $f_x = 0$ and $f_y = 0$ completely, considering every factor.
- Using $D = rt + s^2$ instead of $rt - s^2$.
- Concluding "minimum" when $D = 0$. The test fails; examine f along lines through the point.
- In Lagrange problems, dividing by a variable that could be zero without checking that case separately.
- Writing a Jacobian with rows and columns mixed up. Rows correspond to the functions u, v, w ; columns to the variables x, y, z .

3.15 Solved problems: Module III

Problem 1. If $u = x^3 + y^3 + 3xy^2$, find all first and second order partial derivatives and verify $u_{xy} = u_{yx}$.

$$u_x = 3x^2 + 3y^2, u_y = 3y^2 + 6xy.$$

$$u_{xx} = 6x, u_{yy} = 6y + 6x.$$

$$u_{xy} = \partial/\partial y(3x^2 + 3y^2) = 6y; u_{yx} = \partial/\partial x(3y^2 + 6xy) = 6y.$$

Answer: $u_{xy} = u_{yx} = 6y$ (verified).

Problem 2. If $u = \log(x^2 + y^2)$, show that $u_{xx} + u_{yy} = 0$.

$$u_x = 2x/(x^2 + y^2). u_{xx} = [2(x^2 + y^2) - 2x \cdot 2x]/(x^2 + y^2)^2 = 2(y^2 - x^2)/(x^2 + y^2)^2.$$

$$\text{By symmetry, } u_{yy} = 2(x^2 - y^2)/(x^2 + y^2)^2.$$

$$\text{Adding: } u_{xx} + u_{yy} = 0.$$

Answer: u satisfies Laplace's equation (u is harmonic).

Problem 3. If $u = e^{xyz}$, show that $\partial^3 u / \partial x \partial y \partial z = (1 + 3xyz + x^2 y^2 z^2) e^{xyz}$.

$$u_z = xy e^{xyz}.$$

$$u_{yz} = \partial/\partial y(xy e^{xyz}) = x e^{xyz} + xy \cdot xz e^{xyz} = (x + x^2 yz) e^{xyz}.$$

$$u_{xyz} = \partial/\partial x[(x + x^2yz)e^{xyz}] = (1 + 2xyz)e^{xyz} + (x + x^2yz) yz e^{xyz} = (1 + 2xyz + xyz + x^2y^2z^2)e^{xyz}.$$

Answer: $(1 + 3xyz + x^2y^2z^2) e^{xyz}$.

Problem 4. If $u = f(r)$ where $r^2 = x^2 + y^2$, show that $u_{xx} + u_{yy} = f''(r) + f'(r)/r$.

$$r_x = x/r, u_x = f'(r) x/r.$$

$$u_{xx} = f''(r)(x^2/r^2) + f'(r)[1/r - x^2/r^3].$$

$$\text{Similarly } u_{yy} = f''(r)(y^2/r^2) + f'(r)[1/r - y^2/r^3].$$

$$\text{Adding: } u_{xx} + u_{yy} = f''(r)(x^2 + y^2)/r^2 + f'(r)[2/r - (x^2 + y^2)/r^3] = f''(r) + f'(r)/r.$$

Answer: Proved.

Problem 5. If $u = \sin^{-1}[(x^2 + y^2)/(x + y)]$, show that $x u_x + y u_y = \tan u$.

u itself is not homogeneous, but $z = \sin u = (x^2 + y^2)/(x + y)$ is homogeneous of degree 1.

$$\text{By Euler's theorem: } x z_x + y z_y = z, \text{ i.e. } x \cos u \cdot u_x + y \cos u \cdot u_y = \sin u.$$

Divide by $\cos u$.

Answer: $x u_x + y u_y = \tan u$.

Problem 6. If $z = x^2 + y^2$ and $x = t^2, y = 2t$, find dz/dt .

$$dz/dt = z_x dx/dt + z_y dy/dt = 2x(2t) + 2y(2) = 2t^2(2t) + 4t(2) = 4t^3 + 8t.$$

$$\text{Check: } z = t^4 + 4t^2, dz/dt = 4t^3 + 8t.$$

Answer: $dz/dt = 4t^3 + 8t$.

Problem 7. Find dy/dx if $x^3 + y^3 = 3axy$.

$$f(x, y) = x^3 + y^3 - 3axy. f_x = 3x^2 - 3ay, f_y = 3y^2 - 3ax.$$

Answer: $dy/dx = -f_x/f_y = (ay - x^2)/(y^2 - ax)$.

Problem 8. The period of a simple pendulum is $T = 2\pi\sqrt{l/g}$. Find the maximum percentage error in T due to errors of 1% in l and 2% in g .

$$\log T = \log 2\pi + (1/2)\log l - (1/2)\log g. \text{ Taking differentials: } dT/T = (1/2)(dl/l) - (1/2)(dg/g).$$

$$\text{Maximum error when the signs add: } 100 dT/T = (1/2)(1) + (1/2)(2) = 1.5.$$

Answer: Maximum error in T is 1.5%.

Problem 9. If $x = r \cos \theta, y = r \sin \theta$, find $\partial(x, y)/\partial(r, \theta)$ and $\partial(r, \theta)/\partial(x, y)$.

$$x_r = \cos \theta, x_\theta = -r \sin \theta, y_r = \sin \theta, y_\theta = r \cos \theta.$$

$$J = \cos \theta \cdot r \cos \theta - (-r \sin \theta) \sin \theta = r(\cos^2 \theta + \sin^2 \theta) = r.$$

$$\text{By the property } JJ' = 1, J' = 1/r.$$

Answer: $\partial(x, y)/\partial(r, \theta) = r; \partial(r, \theta)/\partial(x, y) = 1/r$.

Problem 10. If $u = x(1 - y), v = xy$, find $\partial(u, v)/\partial(x, y)$.

$$u_x = 1 - y, u_y = -x, v_x = y, v_y = x.$$

$$J = (1 - y)x - (-x)(y) = x - xy + xy = x.$$

Answer: $\partial(u, v)/\partial(x, y) = x$.

Problem 11. If $u = x + y + z$, $v = y + z$, $w = z$, find $\partial(u, v, w)/\partial(x, y, z)$.

The matrix of partials is rows $(1, 1, 1)$, $(0, 1, 1)$, $(0, 0, 1)$.

It is upper triangular, so the determinant is the product of the diagonal: $1 \times 1 \times 1$.

Answer: $J = 1$.

Problem 12. Show that in spherical coordinates $\partial(x, y, z)/\partial(r, \theta, \phi) = r^2 \sin \theta$.

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta.$$

Row x: $(\sin \theta \cos \phi, r \cos \theta \cos \phi, -r \sin \theta \sin \phi)$. Row y: $(\sin \theta \sin \phi, r \cos \theta \sin \phi, r \sin \theta \cos \phi)$. Row z: $(\cos \theta, -r \sin \theta, 0)$.

Expand along row z: $J = \cos \theta [r \cos \theta \cos \phi \cdot r \sin \theta \cos \phi + r \sin \theta \sin \phi \cdot r \cos \theta \sin \phi] + r \sin \theta [\sin \theta \cos \phi \cdot r \sin \theta \cos \phi + r \sin \theta \sin \phi \cdot \sin \theta \sin \phi]$.

$$= \cos \theta \cdot r^2 \sin \theta \cos \theta (\cos^2 \phi + \sin^2 \phi) + r \sin \theta \cdot r \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) = r^2 \sin \theta (\cos^2 \theta + \sin^2 \theta).$$

Answer: $J = r^2 \sin \theta$.

Problem 13. Show that $u = (x + y)/(1 - xy)$ and $v = \tan^{-1}x + \tan^{-1}y$ are functionally dependent, and find the relation.

$$u_x = [(1 - xy) + y(x + y)]/(1 - xy)^2 = (1 + y^2)/(1 - xy)^2. \text{ By symmetry } u_y = (1 + x^2)/(1 - xy)^2.$$

$$v_x = 1/(1 + x^2), v_y = 1/(1 + y^2).$$

$$J = u_x v_y - u_y v_x = 1/(1 - xy)^2 - 1/(1 - xy)^2 = 0. \text{ So } u \text{ and } v \text{ are functionally dependent.}$$

$$\text{Relation: } \tan v = \tan(\tan^{-1}x + \tan^{-1}y) = (x + y)/(1 - xy) = u.$$

Answer: $J = 0$; $u = \tan v$.

Problem 14. Find the Hessian matrix of $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ at the point $(6, 0)$.

$$f_x = 3x^2 + 3y^2 - 30x + 72, f_y = 6xy - 30y.$$

$$f_{xx} = 6x - 30, f_{xy} = 6y, f_{yy} = 6x - 30.$$

$$\text{At } (6, 0): f_{xx} = 6, f_{xy} = 0, f_{yy} = 6.$$

Answer: $H = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, $\det H = 36 > 0$ with $f_{xx} > 0$ (positive definite).

Problem 15. Find the maximum and minimum values of $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

$$f_x = 3x^2 + 3y^2 - 30x + 72 = 0 \Rightarrow x^2 + y^2 - 10x + 24 = 0 \dots (1). f_y = 6xy - 30y = 6y(x - 5) = 0 \dots (2).$$

$$\text{From (2): } y = 0 \text{ or } x = 5. \text{ If } y = 0: x^2 - 10x + 24 = 0 \Rightarrow x = 4, 6. \text{ If } x = 5: 25 + y^2 - 50 + 24 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1.$$

Stationary points: (4, 0), (6, 0), (5, 1), (5, -1). $r = 6x - 30$, $s = 6y$, $t = 6x - 30$.

(4, 0): $r = -6$, $s = 0$, $t = -6$, $D = 36 > 0$, $r < 0 \Rightarrow$ maximum. $f(4, 0) = 64 - 240 + 288 = 112$.

(6, 0): $r = 6$, $D = 36 > 0$, $r > 0 \Rightarrow$ minimum. $f(6, 0) = 216 - 540 + 432 = 108$.

(5, ± 1): $r = 0$, $s = \pm 6$, $t = 0$, $D = 0 - 36 < 0 \Rightarrow$ saddle points.

Answer: Maximum 112 at (4, 0); minimum 108 at (6, 0); saddle points at (5, ± 1).

Problem 16. Find the extreme values of $f(x, y) = x^3 + y^3 - 3axy$ ($a > 0$).

$$f_x = 3x^2 - 3ay = 0 \Rightarrow y = x^2/a, f_y = 3y^2 - 3ax = 0 \Rightarrow y^2 = ax.$$

Substituting: $x^4/a^2 = ax \Rightarrow x(x^3 - a^3) = 0 \Rightarrow x = 0$ or $x = a$. Points: (0, 0) and (a, a).

$r = 6x$, $s = -3a$, $t = 6y$. At (0, 0): $D = 0 - 9a^2 < 0 \Rightarrow$ saddle point.

At (a, a): $r = 6a > 0$, $D = 36a^2 - 9a^2 = 27a^2 > 0 \Rightarrow$ minimum. $f(a, a) = a^3 + a^3 - 3a^3 = -a^3$.

Answer: Minimum value $-a^3$ at (a, a); (0, 0) is a saddle point.

Problem 17. Find the stationary points of $f(x, y) = x^2 - y^2$ and classify them.

$$f_x = 2x = 0, f_y = -2y = 0 \Rightarrow (0, 0).$$

$r = 2$, $s = 0$, $t = -2$, $D = -4 < 0$.

Answer: (0, 0) is a saddle point (the classic saddle surface).

Problem 18. Find the extreme values of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

$$f_x = 4x^3 - 4x + 4y = 0, f_y = 4y^3 + 4x - 4y = 0. \text{ Adding: } x^3 + y^3 = 0 \Rightarrow y = -x.$$

Then $x^3 - x - x = 0 \Rightarrow x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$. Points: (0, 0), ($\sqrt{2}$, $-\sqrt{2}$), ($-\sqrt{2}$, $\sqrt{2}$).

$r = 12x^2 - 4$, $s = 4$, $t = 12y^2 - 4$.

At ($\pm\sqrt{2}$, $\mp\sqrt{2}$): $r = 20$, $t = 20$, $D = 400 - 16 = 384 > 0$, $r > 0 \Rightarrow$ minimum. $f = 4 + 4 - 4 - 8 - 4 = -8$.

At (0, 0): $r = -4$, $t = -4$, $D = 16 - 16 = 0 \Rightarrow$ test fails. Along $y = 0$: $f = x^4 - 2x^2 < 0$ near 0; along $y = x$: $f = 2x^4 > 0$. So (0, 0) is a saddle point (no extremum).

Answer: Minimum -8 at ($\sqrt{2}$, $-\sqrt{2}$) and ($-\sqrt{2}$, $\sqrt{2}$); no extremum at (0, 0).

Problem 19. Find three positive numbers whose sum is 24 and whose product is maximum (use Lagrange multipliers).

Maximise $f = xyz$ subject to $\phi = x + y + z - 24 = 0$. $F = xyz + \lambda(x + y + z - 24)$.

$$F_x = yz + \lambda = 0, F_y = xz + \lambda = 0, F_z = xy + \lambda = 0 \Rightarrow yz = xz = xy \Rightarrow x = y = z.$$

From the constraint $3x = 24 \Rightarrow x = y = z = 8$.

Answer: The numbers are 8, 8, 8; maximum product = 512.

Problem 20. Find the dimensions of a rectangular box, open at the top, of maximum capacity whose surface area is 432 cm².

Let the sides be x , y and height z . Maximise $V = xyz$ subject to $xy + 2xz + 2yz = 432$.



$$F = xyz + \lambda(xy + 2xz + 2yz - 432). F_x = yz + \lambda(y + 2z) = 0, F_y = xz + \lambda(x + 2z) = 0, F_z = xy + \lambda(2x + 2y) = 0.$$

Multiply the equations by x, y, z respectively: $xyz = -\lambda(xy + 2xz) = -\lambda(xy + 2yz) = -\lambda(2xz + 2yz)$.

From the first two: $2xz = 2yz \Rightarrow x = y$. From the second and third: $xy = 2xz \Rightarrow y = 2z$. So $x = y = 2z$.

$$\text{Constraint: } 4z^2 + 4z^2 + 4z^2 = 12z^2 = 432 \Rightarrow z^2 = 36 \Rightarrow z = 6, x = y = 12.$$

Answer: Dimensions 12 cm $\times$ 12 cm $\times$ 6 cm; maximum volume = 864 cm³.

Problem 21. Find the point on the plane $x + 2y + 3z = 4$ nearest to the origin.

Minimise $f = x^2 + y^2 + z^2$ (square of distance) subject to $x + 2y + 3z = 4$.

$$F = x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 4). 2x + \lambda = 0, 2y + 2\lambda = 0, 2z + 3\lambda = 0 \Rightarrow x = -\lambda/2, y = -\lambda, z = -3\lambda/2.$$

$$\text{Substitute: } -\lambda/2 - 2\lambda - 9\lambda/2 = 4 \Rightarrow -7\lambda = 4 \Rightarrow \lambda = -4/7.$$

$$x = 2/7, y = 4/7, z = 6/7. \text{ Minimum distance} = \sqrt{(4 + 16 + 36)/7} = \sqrt{56/7} = 2\sqrt{14}/7.$$

Answer: Nearest point (2/7, 4/7, 6/7); shortest distance = $4/\sqrt{14} \approx 1.069$.

Problem 22. Find the maximum value of $x^2y^3z^4$ subject to $x + y + z = 5$ (using Lagrange multipliers).

Work with $\log f = 2 \log x + 3 \log y + 4 \log z$ (same maximiser). $F = 2 \log x + 3 \log y + 4 \log z + \lambda(x + y + z - 5)$.

$$2/x + \lambda = 0, 3/y + \lambda = 0, 4/z + \lambda = 0 \Rightarrow x = -2/\lambda, y = -3/\lambda, z = -4/\lambda. \text{ Sum: } -9/\lambda = 5 \Rightarrow \lambda = -9/5.$$

$$x = 10/9, y = 15/9 = 5/3, z = 20/9.$$

Answer: Maximum at (10/9, 5/3, 20/9); value = $(10/9)^2(5/3)^3(20/9)^4 \approx 139.4$.

Problem 23. Find the maximum and minimum values of $x^2 + y^2$ subject to $x^2 + xy + y^2 = 3$.

$$F = x^2 + y^2 + \lambda(x^2 + xy + y^2 - 3). F_x = 2x + \lambda(2x + y) = 0, F_y = 2y + \lambda(x + 2y) = 0.$$

$$\text{From these, } -\lambda = 2x/(2x + y) = 2y/(x + 2y) \Rightarrow 2x(x + 2y) = 2y(2x + y) \Rightarrow x^2 = y^2 \Rightarrow y = \pm x.$$

$$y = x: 3x^2 = 3 \Rightarrow x = \pm 1, f = 2. y = -x: x^2 = 3 \Rightarrow f = 6.$$

Answer: Maximum of $x^2 + y^2$ is 6 (at $(\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$); minimum is 2 (at $(1, 1), (-1, -1)$).

3.16 Practice problems with solutions: Module III

Q1. If $z = x^2y + y^3$, find z_x, z_y and z_{xy} .

$$z_x = 2xy; z_y = x^2 + 3y^2; z_{xy} = \partial/\partial y(2xy) = 2x.$$

Answer: $2xy, x^2 + 3y^2, 2x$.

Q2. Show that $u = e^x \cos y$ is harmonic.



$$u_{xx} = e^x \cos y; u_{yy} = -e^x \cos y.$$

Answer: $u_{xx} + u_{yy} = 0$, so u is harmonic.

Q3. If $u = x^3 + y^3 + z^3 - 3xyz$, find $u_x + u_y + u_z$.

$$u_x = 3x^2 - 3yz, u_y = 3y^2 - 3xz, u_z = 3z^2 - 3xy.$$

Answer: $3(x^2 + y^2 + z^2 - xy - yz - zx)$.

Q4. If $u = \tan^{-1}(y/x)$, verify $u_{xy} = u_{yx}$.

$$u_x = -y/(x^2 + y^2), u_y = x/(x^2 + y^2).$$

$$u_{xy} = \partial/\partial y[-y/(x^2 + y^2)] = [-(x^2 + y^2) + 2y^2]/(x^2 + y^2)^2 = (y^2 - x^2)/(x^2 + y^2)^2.$$

$$u_{yx} = \partial/\partial x[x/(x^2 + y^2)] = [(x^2 + y^2) - 2x^2]/(x^2 + y^2)^2 = (y^2 - x^2)/(x^2 + y^2)^2.$$

Answer: Both equal $(y^2 - x^2)/(x^2 + y^2)^2$.

Q5. Verify Euler's theorem for $u = x^4 + y^4 + x^2y^2$.

$$u \text{ is homogeneous of degree } 4. u_x = 4x^3 + 2xy^2, u_y = 4y^3 + 2x^2y.$$

$$x u_x + y u_y = 4x^4 + 2x^2y^2 + 4y^4 + 2x^2y^2 = 4(x^4 + y^4 + x^2y^2).$$

Answer: $x u_x + y u_y = 4u$. Verified.

Q6. If $u = x + y, v = x - y$, find $\partial(u, v)/\partial(x, y)$.

$$J = (1)(-1) - (1)(1).$$

Answer: -2 .

Q7. If $x = u(1 + v), y = v(1 + u)$, find $\partial(x, y)/\partial(u, v)$.

$$x_u = 1 + v, x_v = u, y_u = v, y_v = 1 + u.$$

$$J = (1 + v)(1 + u) - uv = 1 + u + v.$$

Answer: $1 + u + v$.

Q8. Show that $u = x + y, v = x^2 + 2xy + y^2$ are functionally dependent and find the relation.

$$J = \begin{vmatrix} 1 & 1 \\ 2x + 2y & 2x + 2y \end{vmatrix} = (2x + 2y) - (2x + 2y) = 0 \Rightarrow \text{dependent.}$$

Answer: $v = (x + y)^2 = u^2$.

Q9. Find the extreme value of $f = x^2 + xy + y^2 + 3x - 3y + 4$.

$$f_x = 2x + y + 3 = 0, f_y = x + 2y - 3 = 0 \Rightarrow x = -3, y = 3.$$

$$r = 2, s = 1, t = 2, D = 3 > 0, r > 0 \Rightarrow \text{minimum.}$$

Answer: Minimum $f(-3, 3) = 9 - 9 + 9 - 9 - 9 + 4 = -5$.

Q10. Find the extreme values of $f = 2(x^2 - y^2) - x^4 + y^4$.

$$f_x = 4x - 4x^3 = 0 \Rightarrow x = 0, \pm 1; f_y = -4y + 4y^3 = 0 \Rightarrow y = 0, \pm 1. r = 4 - 12x^2, s = 0, t = -4 + 12y^2.$$

$$(\pm 1, 0): r = -8, t = -4, D = 32 > 0, r < 0 \Rightarrow \text{max, } f = 2 - 1 = 1.$$



$(0, \pm 1)$: $r = 4$, $t = 8$, $D = 32 > 0$, $r > 0 \Rightarrow \min$, $f = -2 + 1 = -1$.

$(0, 0)$: $r = 4$, $t = -4$, $D < 0 \Rightarrow \text{saddle}$. $(\pm 1, \pm 1)$: $r = -8$, $t = 8$, $D < 0 \Rightarrow \text{saddle}$.

Answer: Max 1 at $(\pm 1, 0)$; min -1 at $(0, \pm 1)$; saddle points at $(0, 0)$ and $(\pm 1, \pm 1)$.

Q11. Find the extreme values of $f = xy(a - x - y)$, $a > 0$.

$f_x = y(a - 2x - y) = 0$, $f_y = x(a - x - 2y) = 0$. Solutions: $(0, 0)$, $(a, 0)$, $(0, a)$, $(a/3, a/3)$.
 $r = -2y$, $s = a - 2x - 2y$, $t = -2x$. At $(a/3, a/3)$: $r = -2a/3$, $s = -a/3$, $t = -2a/3$, $D = 4a^2/9 - a^2/9 = a^2/3 > 0$, $r < 0 \Rightarrow \max$.

At the other three points $D = -a^2 < 0 \Rightarrow \text{saddle points}$.

Answer: Maximum $a^3/27$ at $(a/3, a/3)$.

Q12. Find the minimum of $x^2 + y^2 + z^2$ subject to $x + y + z = 3a$.

$F = x^2 + y^2 + z^2 + \lambda(x + y + z - 3a) \Rightarrow 2x + \lambda = 2y + \lambda = 2z + \lambda = 0 \Rightarrow x = y = z = a$.

Answer: Minimum $3a^2$.

Q13. Find the largest rectangular parallelepiped that can be inscribed in the sphere $x^2 + y^2 + z^2 = a^2$.

$V = 8xyz$. $F = 8xyz + \lambda(x^2 + y^2 + z^2 - a^2)$. $8yz + 2\lambda x = 0$ etc. Multiplying by x, y, z : $x^2 = y^2 = z^2 \Rightarrow x = y = z = a/\sqrt{3}$.

Answer: A cube of side $2a/\sqrt{3}$; volume $8a^3/(3\sqrt{3})$.

Q14. Find the maximum of xyz subject to $xy + yz + zx = 12$ ($x, y, z > 0$).

$F = xyz + \lambda(xy + yz + zx - 12)$: $yz + \lambda(y + z) = 0$, $xz + \lambda(x + z) = 0$, $xy + \lambda(x + y) = 0$.

Multiply by x, y, z and compare: $xy + xz = xy + yz = xz + yz \Rightarrow x = y = z$. Then $3x^2 = 12 \Rightarrow x = 2$.

Answer: Maximum 8 at $x = y = z = 2$.

Q15. Find the equation of the tangent plane to $z = x^2 + y^2$ at $(1, 2, 5)$.

$f_x = 2x = 2$, $f_y = 2y = 4$ at $(1, 2)$.

Answer: $z - 5 = 2(x - 1) + 4(y - 2)$, i.e. $2x + 4y - z = 5$.

3.17 MCQs: Module III

1. If $z = x^2y$, then $\partial z / \partial x$ is

- (a) $2xy$ (b) x^2y (c) x^2 (d) $2x$

□ (a) — Treat y as constant.

2. If $z = x^2y$, then $\partial^2 z / \partial x \partial y$ is

- (a) x^2 (b) $2x$ (c) 2 (d) $2y$

□ (b) — $\partial / \partial x (x^2) = 2x$.

3. If $f = e^x \sin y$, then $f_{xx} + f_{yy}$ equals

- (a) 0 (b) 1 (c) $2e^x \sin y$ (d) e^x

□ (a) — $f_{xx} = e^x \sin y$, $f_{yy} = -e^x \sin y$.



4. A function satisfying $u_{xx} + u_{yy} = 0$ is called
(a) harmonic (b) homogeneous (c) analytic (d) periodic
□ (a) — Laplace's equation.
5. Clairaut's theorem states that, under continuity,
(a) $f_{xy} = f_{yx}$ (b) $f_{xy} = f_{yx}$ (c) $f_{xx} = f_{yy}$ (d) $f_{xy} = 0$
□ (b) — Equality of mixed partials.
6. If $f(x, y)$ is homogeneous of degree n , then $x f_x + y f_y$ equals
(a) nf (b) 0 (c) $n(n-1)f$ (d) f
□ (a) — Euler's theorem.
7. If $u = x^2/y$ (homogeneous degree 1), then $x u_x + y u_y$ equals
(a) u (b) $-u$ (c) $2u$ (d) 0
□ (a) — Degree 1.
8. The degree of the homogeneous function $(x^3 + y^3)/(x + y)$ is
(a) 4 (b) 3 (c) 2 (d) 1
□ (c) — $3 - 1 = 2$.
9. If $u = f(y/x)$, then $x u_x + y u_y$ equals
(a) 1 (b) u (c) 0 (d) y/x
□ (c) — Homogeneous of degree 0.
10. For $z = f(x, y)$ with $x = x(t)$, $y = y(t)$, dz/dt equals
(a) $z_x + z_y$ (b) $z_x x' + z_y y'$ (c) $z_x z_y$ (d) $z_x y' + z_y x'$
□ (b) — Chain rule.
11. For $f(x, y) = 0$, dy/dx equals
(a) f_x/f_y (b) $-f_x/f_y$ (c) $-f_y/f_x$ (d) f_y/f_x
□ (b) — Implicit differentiation.
12. The Jacobian $\partial(u, v)/\partial(x, y)$ equals
(a) $u_x v_y - u_y v_x$ (b) $u_x v_y + u_y v_x$ (c) $u_x u_y - v_x v_y$ (d) $u_x v_y - u_y v_x$
□ (d) — 2×2 determinant.
13. For $x = r \cos \theta$, $y = r \sin \theta$, $\partial(x, y)/\partial(r, \theta)$ equals
(a) r^2 (b) $1/r$ (c) 1 (d) r
□ (d) — Polar Jacobian.
14. For spherical coordinates, $\partial(x, y, z)/\partial(r, \theta, \phi)$ equals
(a) $r \sin \theta$ (b) $r^2 \sin \theta$ (c) r (d) r^2
□ (b) — Standard result.
15. For cylindrical coordinates, $\partial(x, y, z)/\partial(r, \theta, z)$ equals
(a) $r \sin \theta$ (b) r (c) 1 (d) r^2
□ (b) — Same as polar.
16. If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$, then JJ' equals
(a) 1 (b) 0 (c) -1 (d) J^2
□ (a) — Inverse property.
17. u and v are functionally dependent if $\partial(u, v)/\partial(x, y)$ is
(a) negative (b) 0 identically (c) positive (d) 1
□ (b) — Jacobian test.
18. If $u = x + y$, $v = x - y$, then $\partial(u, v)/\partial(x, y)$ equals
(a) 1 (b) 2 (c) 0 (d) -2
□ (d) — $1 \cdot (-1) - 1 \cdot 1$.



- 19.** If $u = 2x + 3y$, $v = x - y$, then $\partial(u, v)/\partial(x, y)$ equals
(a) 1 (b) -1 (c) 5 (d) -5
□ (d) $-2(-1) - 3(1)$.
- 20.** The Hessian matrix of a function $f(x, y)$ is always
(a) singular (b) symmetric (c) diagonal (d) skew-symmetric
□ (b) $-f_{xy} = f_{yx}$.
- 21.** The Hessian matrix consists of
(a) mixed derivatives only (b) Jacobians (c) first partial derivatives (d) second partial derivatives
□ (d) — Matrix of f_{xx}, f_{xy}, f_{yy} .
- 22.** At a stationary point of $f(x, y)$,
(a) $f_x = 0$ only (b) $f_x = 0$ and $f_y = 0$ (c) $f_{xx} = 0$ (d) $f_y = 0$ only
□ (b) — Necessary condition.
- 23.** With $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$, f has a local maximum if
(a) $rt - s^2 > 0$, $r < 0$ (b) $rt - s^2 > 0$, $r > 0$ (c) $rt - s^2 = 0$ (d) $rt - s^2 < 0$
□ (a) — Lagrange's conditions.
- 24.** f has a local minimum if
(a) $rt - s^2 > 0$, $r > 0$ (b) $rt - s^2 > 0$, $r < 0$ (c) $rt - s^2 < 0$ (d) $r = 0$
□ (a) — Positive definite Hessian.
- 25.** If $rt - s^2 < 0$ at a stationary point, the point is
(a) maximum (b) undecided (c) minimum (d) saddle point
□ (d) — Indefinite Hessian.
- 26.** If $rt - s^2 = 0$ at a stationary point, then
(a) it is a minimum (b) it is a maximum (c) it is a saddle point (d) further investigation is needed
□ (d) — Test fails.
- 27.** The stationary point of $f(x, y) = x^2 + y^2$ is
(a) (1, 1) (b) (1, 0) (c) (0, 0) (d) none
□ (c) $-f_x = 2x = 0$, $f_y = 2y = 0$.
- 28.** The point (0, 0) for $f(x, y) = x^2 - y^2$ is a
(a) saddle point (b) maximum (c) minimum (d) not stationary
□ (a) $-D = (2)(-2) - 0 = -4 < 0$.
- 29.** The minimum value of $f(x, y) = x^2 + y^2 + 6x + 12$ is
(a) 3 (b) 0 (c) -3 (d) 12
□ (a) — At $(-3, 0)$: $9 - 18 + 12 = 3$.
- 30.** For $f = x^3 + y^3 - 3xy$, the stationary points are
(a) (1, 0) and (0, 1) (b) (0, 0) only (c) (0, 0) and (1, 1) (d) (1, 1) only
□ (c) $-x^2 = y$, $y^2 = x$.
- 31.** For $f = x^3 + y^3 - 3xy$, the point (1, 1) is a
(a) minimum (b) saddle point (c) maximum (d) none
□ (a) $-D = 36 - 9 > 0$, $r = 6 > 0$.
- 32.** The saddle point of $f = x^3 + y^3 - 3xy$ is
(a) (1, 0) (b) (1, 1) (c) (-1, -1) (d) (0, 0)
□ (d) $-D = -9 < 0$.
- 33.** In the Lagrange multiplier method, the Lagrangian is
(a) $f - \varphi$ (b) $f\varphi$ (c) f/φ (d) $f + \lambda\varphi$
□ (d) $-F = f + \lambda\varphi$.



- 34.** The Lagrange multiplier method is used to find extrema of
(a) only one-variable functions (b) only linear functions (c) functions subject to constraints (d) functions with no constraints
□ (c) — Constrained optimisation.
- 35.** At a constrained extremum, ∇f and $\nabla \phi$ are
(a) parallel (b) unrelated (c) zero (d) perpendicular
□ (a) — $\nabla f = -\lambda \nabla \phi$.
- 36.** The maximum of xy subject to $x + y = 10$ is
(a) 50 (b) 20 (c) 100 (d) 25
□ (d) — $x = y = 5$.
- 37.** The minimum of $x^2 + y^2$ subject to $x + y = 2$ is
(a) 2 (b) 0 (c) 4 (d) 1
□ (a) — $x = y = 1$.
- 38.** The maximum product of three positive numbers whose sum is 12 is
(a) 36 (b) 48 (c) 64 (d) 27
□ (c) — $4 \times 4 \times 4$.
- 39.** An open rectangular box of maximum volume with given surface area has
(a) base square and height half the side (b) height equal to side (c) base square and height double the side (d) all sides equal
□ (a) — $x = y = 2z$.
- 40.** The shortest distance from the origin to the plane $x + y + z = 3$ is
(a) $1/\sqrt{3}$ (b) 1 (c) 3 (d) $\sqrt{3}$
□ (d) — Point (1, 1, 1).
- 41.** If $f(x, y, z)$ has Hessian leading minors $D_1 > 0, D_2 > 0, D_3 > 0$, the stationary point is
a
(a) undecided (b) saddle point (c) maximum (d) minimum
□ (d) — Positive definite.
- 42.** The total differential of $z = f(x, y)$ is
(a) $dz = z_x dx + z_y dy$ (b) $dz = dx + dy$ (c) $dz = z_x + z_y$ (d) $dz = z_x dy + z_y dx$
□ (a) — Definition.
- 43.** If $z = xy$, the percentage error in z when x and y each have 1% error is at most
(a) 2% (b) 0.5% (c) 1% (d) 4%
□ (a) — $dz/z = dx/x + dy/y$.
- 44.** $\lim_{(x,y) \rightarrow (0,0)} xy/(x^2 + y^2)$
(a) does not exist (b) is 1 (c) is $1/2$ (d) is 0
□ (a) — Depends on path $y = mx$.
- 45.** If $u = \log(x^2 + y^2)$, then u_x is
(a) $2y/(x^2 + y^2)$ (b) $x/(x^2 + y^2)$ (c) $2x/(x^2 + y^2)$ (d) $1/(x^2 + y^2)$
□ (c) — Chain rule.
- 46.** If $u = \tan^{-1}(y/x)$, then $x u_x + y u_y$ equals
(a) u (b) 0 (c) 1 (d) $\tan u$
□ (b) — Homogeneous of degree 0.
- 47.** The Jacobian of $u = x + y + z, v = y + z, w = z$ with respect to x, y, z is
(a) 0 (b) 1 (c) -1 (d) 3
□ (b) — Triangular determinant.



48. The Jacobian used in the transformation to polar coordinates changes $dx\,dy$ into

- (a) $r\,dr\,d\theta$ (b) $dr\,d\theta$ (c) $dr\,d\theta/r$ (d) $r^2\,dr\,d\theta$

□ (a) — $|J| = r$.

49. For $f(x, y)$ with $rt - s^2 > 0$ at a point, which is true?

- (a) r and t have the same sign (b) $s = 0$ (c) r and t have opposite signs (d) $r = 0$

□ (a) — $rt > s^2 \geq 0$.

50. If $u = x\,y\,z$, then $\partial^3 u / \partial x \partial y \partial z$ equals

- (a) xyz (b) $x + y + z$ (c) 0 (d) 1

□ (d) — Differentiate each variable once.

Module IV: Linear Algebra I (Vector Spaces, Linear Systems, Rank)

Linear algebra is the mathematics of systems of linear equations and of vectors. Electrical networks (Kirchhoff's laws), structural analysis, computer graphics, machine learning and signal processing all reduce to equations of the form $AX = B$. This module builds the language (vector spaces, basis, dimension, rank) and the main solution method (Gauss elimination).

4.1 Matrices: basic definitions

- A **matrix** of order $m \times n$ is a rectangular array of numbers with m rows and n columns: $A = [a_{ij}]$.
- **Square matrix**: $m = n$. **Row matrix**: $1 \times n$. **Column matrix**: $m \times 1$. **Zero matrix**: all entries 0.
- **Diagonal matrix**: $a_{ij} = 0$ for $i \neq j$. **Scalar matrix**: diagonal with equal entries. **Identity matrix I**: diagonal entries 1.
- **Upper triangular**: $a_{ij} = 0$ for $i > j$. **Lower triangular**: $a_{ij} = 0$ for $i < j$.
- **Transpose A^T** : rows become columns, $(A^T)_{ij} = a_{ji}$. $(AB)^T = B^T A^T$.
- **Trace**: sum of diagonal entries. $\text{tr}(A + B) = \text{tr} A + \text{tr} B$, $\text{tr}(AB) = \text{tr}(BA)$.
- **Determinant** (square matrices only): $|AB| = |A||B|$, $|A^T| = |A|$, $|kA| = k^n |A|$ for $n \times n$ A .
- **Singular matrix**: $|A| = 0$. **Non-singular**: $|A| \neq 0$.
- Matrix multiplication is associative and distributive but **not commutative** in general ($AB \neq BA$).

4.2 Vector spaces

A **vector space** V over the real numbers R is a non-empty set of objects (called vectors) with two operations, vector addition and scalar multiplication, satisfying the following axioms for all u, v, w in V and scalars a, b :

- **Closure**: $u + v \in V$ and $au \in V$.
- **Addition**: $u + v = v + u$ (commutative); $(u + v) + w = u + (v + w)$ (associative); there is a zero vector 0 with $u + 0 = u$; each u has a negative $-u$ with $u + (-u) = 0$.
- **Scalar multiplication**: $a(u + v) = au + av$; $(a + b)u = au + bu$; $(ab)u = a(bu)$; $1 \cdot u = u$.

Examples of vector spaces.

- R^n : all n -tuples $(x_1, \dots, x_n)$ with component-wise operations.
- $M_{m \times n}$: all $m \times n$ real matrices.
- P_n : all polynomials of degree $\leq n$.
- $C[a, b]$: all continuous functions on $[a, b]$.
- The solution set of a homogeneous system $AX = 0$.

Non-examples. The set of vectors (x, y) with $x + y = 1$ is not a vector space (it does not contain 0 ; not closed under addition). The set of polynomials of degree exactly 2 is not a vector space ($x^2 + (-x^2) = 0$ has degree less than 2).

Subspace. A non-empty subset W of V is a subspace if it is itself a vector space under the same operations. **Test:** W is a subspace iff (i) $0 \in W$, (ii) $u, v \in W \Rightarrow u + v \in W$, (iii) $u \in W \Rightarrow au \in W$. Equivalently, $au + bv \in W$ for all $u, v \in W$ and scalars a, b .



- Lines and planes through the origin are subspaces of R^3 ; a line not through the origin is not.
- $\{(x, y, z) : x + 2y - z = 0\}$ is a subspace of R^3 ; $\{(x, y, z) : x + 2y - z = 5\}$ is not.

4.3 Linear combination, span, linear dependence and independence

- A **linear combination** of $v_1, \dots, v_k$ is any vector $c_1v_1 + c_2v_2 + \dots + c_kv_k$.
- The **span** of a set S is the set of all linear combinations of vectors in S . $\text{Span}(S)$ is always a subspace.
- Vectors $v_1, \dots, v_k$ are **linearly independent (LI)** if $c_1v_1 + \dots + c_kv_k = 0$ implies $c_1 = c_2 = \dots = c_k = 0$.
- They are **linearly dependent (LD)** if there exist scalars, not all zero, with $c_1v_1 + \dots + c_kv_k = 0$. Then at least one vector is a linear combination of the others.

Facts (useful for MCQs):

- Any set containing the zero vector is LD.
- Two vectors are LD iff one is a scalar multiple of the other.
- In R^n , more than n vectors are always LD.
- n vectors in R^n are LI iff the determinant of the matrix formed by them is non-zero.
- k vectors are LI iff the rank of the matrix formed by them (as rows) is k .

Working rule to test LI/LD. Write the vectors as rows (or columns) of a matrix A .

Reduce to echelon form. If $\text{rank } A = \text{number of vectors}$, they are LI; if $\text{rank } A < \text{number of vectors}$, they are LD. To find the relation, solve $c_1v_1 + \dots + c_kv_k = 0$.

4.4 Basis and dimension

- A set $B = \{v_1, \dots, v_n\}$ is a **basis** of V if (i) B is linearly independent and (ii) B spans V .
- Every vector in V can then be written **uniquely** as a linear combination of the basis vectors. The coefficients are the **coordinates** of the vector relative to B .
- All bases of V have the same number of vectors; this number is the **dimension** of V , written $\dim V$.

Standard bases and dimensions:

Vector space	Standard basis	Dimension
R^2	$e_1 = (1, 0), e_2 = (0, 1)$	2
R^3	$(1, 0, 0), (0, 1, 0), (0, 0, 1)$	3
R^n	$e_1, \dots, e_n$	n
P_n (polynomials of degree $\leq n$)	$1, x, x^2, \dots, x^n$	$n + 1$
$M_{2 \times 2}$	four matrices $E_{11}, E_{12}, E_{21}, E_{22}$	4
$M_{m \times n}$	mn matrices E_{ij}	mn
$\{0\}$	empty set	0

Key results. If $\dim V = n$: any n LI vectors form a basis; any n vectors that span V form a basis; any set of more than n vectors is LD; any set of fewer than n vectors cannot span V .

Row space, column space and null space. For an $m \times n$ matrix A : the **row space** is the span of its rows (subspace of R^n); the **column space** is the span of its columns (subspace of R^m); the **null space** is the set of solutions of $AX = 0$ (subspace of R^n).

$\dim(\text{row space}) = \dim(\text{column space}) = \text{rank } A$.

Rank-nullity theorem: $\text{rank}(A) + \text{nullity}(A) = n$ (number of columns), where $\text{nullity} = \dim(\text{null space})$

4.5 Elementary row operations and echelon form

Elementary row operations (they do not change the rank or the solution set of a system):

- $R_i \leftrightarrow R_j$ (interchange two rows).
- $R_i \rightarrow kR_i$, $k \neq 0$ (multiply a row by a non-zero constant).
- $R_i \rightarrow R_i + kR_j$ (add a multiple of one row to another).

Row echelon form (REF). A matrix is in REF if (i) all zero rows are at the bottom, (ii) the first non-zero entry (pivot/leading entry) of each row is to the right of the pivot of the row above, (iii) all entries below a pivot are zero.

Reduced row echelon form (RREF). REF, plus every pivot is 1 and is the only non-zero entry in its column. RREF is unique.

4.6 Rank of a matrix

Definition 1 (minors). The rank of A is r if A has at least one non-zero minor of order r and every minor of order $r + 1$ is zero. Written $\rho(A)$ or $\text{rank } A$.

Definition 2 (echelon form). The rank of A = the number of non-zero rows in its row echelon form. This is the practical method.

Properties.

$\text{rank}(A) \leq \min(m, n)$ $\text{rank}(A) = \text{rank}(A^T)$ $\text{rank}(O) = 0$ $\text{rank}(I_n) = n$
For $n \times n$ A : $\text{rank } A = n \Leftrightarrow |A| \neq 0 \Leftrightarrow A$ is invertible $\Leftrightarrow$ rows (columns) are LI
 $\text{rank}(AB) \leq \min(\text{rank } A, \text{rank } B)$

Normal form. Using both row and column operations, every non-zero $m \times n$ matrix can be reduced to one of the forms $[I_r]$, $[I_r \ 0]$, $[I_r; \ 0]$, or $[I_r \ 0; \ 0 \ 0]$. Then $\text{rank } A = r$.

4.7 Systems of linear equations

A system of m equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \dots, a_{m1}x_1 + \dots + a_{mn}x_n = b_m \text{ i.e. } AX = B$$

Here A is the **coefficient matrix**, X the column of unknowns, B the column of constants, and $[A \mid B]$ the **augmented matrix**.

- If $B = 0$, the system is **homogeneous**; otherwise **non-homogeneous**.
- A system is **consistent** if it has at least one solution, **inconsistent** if it has none.

Rouché-Capelli theorem (consistency test). Let $r = \text{rank } A$, $r' = \text{rank } [A \mid B]$, $n =$ number of unknowns.

$r \neq r' \Rightarrow$ INCONSISTENT (no solution)
 $r = r' = n \Rightarrow$ CONSISTENT with a UNIQUE solution
 $r = r' < n \Rightarrow$ CONSISTENT with INFINITELY MANY solutions $((n - r)$ free parameters)



Homogeneous system $AX = 0$. It is always consistent ($X = 0$ is the trivial solution).

- If rank $A = n$ (for square A : $|A| \neq 0$), only the trivial solution exists.
- If rank $A < n$ (for square A : $|A| = 0$), there are infinitely many non-trivial solutions, forming a subspace of dimension $n - r$.

Geometric view (3 unknowns). Each equation is a plane. A unique solution means three planes meet at a single point; infinitely many solutions means they share a line (or plane); no solution means there is no common point (for example two parallel planes).

4.8 Gauss elimination method

Idea. Use elementary row operations on the augmented matrix $[A | B]$ to reduce A to upper triangular (echelon) form, then solve by **back substitution**.

Steps.

- Write the augmented matrix $[A | B]$.
- Make the first pivot non-zero (swap rows if needed). Use it to create zeros below it in column 1: $R_i \rightarrow R_i - (a_{i1}/a_{11})R_1$.
- Move to the second row and second column; create zeros below the pivot. Continue until the matrix is upper triangular.
- Check consistency using ranks.
- Solve the last equation first, then substitute upwards.

Partial pivoting. In numerical work, swap rows so the pivot is the entry with the largest absolute value in its column. This reduces round-off error and avoids division by zero.

Gauss-Jordan method. Continue the elimination to make zeros above the pivots as well (RREF), so the solution can be read directly without back substitution. (Used for inverses in Module V.)

Operation count. Gauss elimination for an $n \times n$ system needs about $n^3/3$ multiplications, far fewer than Cramer's rule for large n .

4.9 Worked illustration of Gauss elimination with partial pivoting

Solve $2x + y + z = 10$, $3x + 2y + 3z = 18$, $x + 4y + 9z = 16$ using partial pivoting.

- **Column 1:** the largest $|a_{i1}|$ is 3 (row 2), so swap $R_1 \leftrightarrow R_2$: $[3 \ 2 \ 3 \ | \ 18; 2 \ 1 \ 1 \ | \ 10; 1 \ 4 \ 9 \ | \ 16]$.
- Eliminate: $R_2 \rightarrow R_2 - (2/3)R_1 = [0 \ -1/3 \ -1 \ | \ -2]$; $R_3 \rightarrow R_3 - (1/3)R_1 = [0 \ 10/3 \ 8 \ | \ 10]$.
- **Column 2:** compare $|-1/3|$ and $|10/3|$; the larger is $10/3$ (row 3), so swap $R_2 \leftrightarrow R_3$.
- $R_3 \rightarrow R_3 + (1/10)R_2 = [0 \ 0 \ -1 + 0.8 \ | \ -2 + 1] = [0 \ 0 \ -0.2 \ | \ -1] \Rightarrow z = 5$.
- Back substitution: $(10/3)y + 8(5) = 10 \Rightarrow y = -9$; $3x + 2(-9) + 3(5) = 18 \Rightarrow x = 7$.

The answer $(7, -9, 5)$ agrees with the earlier solution. Pivoting did not change the answer; it only keeps the multipliers small ($|m| \leq 1$), which limits the growth of round-off error on a computer.

4.10 Why elementary row operations do not change the solution

Each elementary row operation replaces the system by an equivalent one:

- Swapping two equations obviously changes nothing.
- Multiplying an equation by $k \neq 0$ does not change its solution set (divide by k to get back).

- Adding k times equation j to equation i : any solution of the old system satisfies the new one, and the operation can be undone by subtracting k times equation j .

Each row operation corresponds to multiplying on the left by an invertible **elementary matrix** E (obtained by doing that operation on I). Since $EAX = EB \Leftrightarrow AX = B$ when E is invertible, the solution set is unchanged. Because E is invertible, $\text{rank}(EA) = \text{rank } A$; hence row operations do not change the rank either.

4.11 Geometry of solution sets

- In $\mathbb{R}^2$, each linear equation $ax + by = c$ is a line. Two equations: intersecting lines (unique solution), parallel distinct lines (no solution), or the same line (infinitely many solutions).
- In $\mathbb{R}^3$, each equation is a plane. Three planes can meet in a single point (rank 3), along a common line (rank 2, consistent), coincide (rank 1, consistent), or have no common point (inconsistent). No common point happens when two planes are parallel, or when the three planes form a triangular prism.
- The general solution of a consistent system has the form $X = X_p + X_h$, where X_p is one particular solution of $AX = B$ and X_h is the general solution of $AX = 0$ (null space). So the solution set is the null space shifted to pass through X_p .

4.12 Row space, column space and null space: an example

Let $A = [1 \ 2 \ 3; 2 \ 4 \ 6; 1 \ 3 \ 4]$.

- Echelon form: $R_2 - 2R_1 = 0$; $R_3 - R_1 = (0, 1, 1)$. After swapping, the non-zero rows are $(1, 2, 3)$, $(0, 1, 1)$. So $\text{rank } A = 2$.
- **Row space**: $\text{span}\{(1, 2, 3), (0, 1, 1)\}$, dimension 2.
- **Column space**: the pivot columns of the original A (columns 1 and 2) form a basis: $\{(1, 2, 1), (2, 4, 3)\}$, dimension 2.
- **Null space**: from $y + z = 0$ and $x + 2y + 3z = 0$, with $z = t$: $y = -t$, $x = -t$. Basis $\{(-1, -1, 1)\}$, nullity 1.
- Check: $\text{rank} + \text{nullity} = 2 + 1 = 3 = \text{number of columns}$ $\square$.

4.13 Linear transformations (connection to matrices)

A map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **linear** if $T(u + v) = T(u) + T(v)$ and $T(cu) = cT(u)$. Every linear map is multiplication by an $m \times n$ matrix A : $T(X) = AX$, where the columns of A are $T(e_1)$, ..., $T(e_n)$.

- The **kernel** (null space) of T is $\{X : AX = 0\}$; T is one-to-one iff the kernel is $\{0\}$.
- The **range** (image) of T is the column space of A ; its dimension is $\text{rank } A$.
- Examples in $\mathbb{R}^2$: rotation by θ has matrix $[\cos \theta \ -\sin \theta; \sin \theta \ \cos \theta]$; reflection in the x -axis has matrix $[1 \ 0; 0 \ -1]$; projection onto the x -axis has matrix $[1 \ 0; 0 \ 0]$ (rank 1, nullity 1).

4.14 Engineering applications of linear systems

- **Electrical networks**: Kirchhoff's current and voltage laws give linear equations in branch currents. For example, mesh analysis of a three-mesh circuit gives a 3×3 system that is solved by Gauss elimination (see Basic Electrical Engineering, Module I).
- **Structural analysis**: equilibrium of joints in a truss gives linear equations in member forces (Engineering Mechanics, method of joints).



- **Chemical equation balancing:** balancing $C_3H_8 + O_2 \rightarrow CO_2 + H_2O$ leads to a homogeneous system; its non-trivial solution gives the coefficients 1, 5, 3, 4.
- **Computer graphics:** rotations, scaling and projections are matrix multiplications.
- **Data fitting:** least-squares fitting of a straight line leads to a 2×2 linear system (normal equations).

4.15 Common mistakes to avoid in Module IV

- Applying a row operation to the coefficient matrix but forgetting the right-hand column of the augmented matrix.
- Declaring a system inconsistent because $|A| = 0$. A zero determinant only means there is no unique solution; the system may still have infinitely many solutions.
- Counting a row like $(0 \ 0 \ 0 \mid 5)$ as a zero row. It means $0 = 5$, so the system is inconsistent.
- Confusing "LI vectors" with "basis". A basis must also span the space.
- Forgetting that a subspace must contain the zero vector.

4.16 Solved problems: Module IV

Problem 1. Show that $W = \{(x, y, z) \in \mathbb{R}^3 : x - 2y + 3z = 0\}$ is a subspace of $\mathbb{R}^3$. Find a basis and its dimension.

$(0, 0, 0)$ satisfies the equation, so $0 \in W$.

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in W$ and a, b scalars. Then $(ax_1 + bx_2) - 2(ay_1 + by_2) + 3(az_1 + bz_2) = a(0) + b(0) = 0$. So $au + bv \in W$. Hence W is a subspace.

Solving $x = 2y - 3z$ with y, z free: $(x, y, z) = y(2, 1, 0) + z(-3, 0, 1)$.

The vectors $(2, 1, 0)$ and $(-3, 0, 1)$ span W and are LI (neither is a multiple of the other).

Answer: W is a subspace (a plane through the origin); basis $\{(2, 1, 0), (-3, 0, 1)\}$; $\dim W = 2$.

Problem 2. Is $W = \{(x, y) : y = 2x + 1\}$ a subspace of $\mathbb{R}^2$?

$(0, 0)$: $0 \neq 2(0) + 1$, so the zero vector is not in W .

Answer: No, W is not a subspace (it is a line not through the origin).

Problem 3. Is $W = \{(x, y, z) : x \geq 0\}$ a subspace of $\mathbb{R}^3$?

$u = (1, 0, 0) \in W$ but $(-1)u = (-1, 0, 0) \notin W$. Not closed under scalar multiplication.

Answer: Not a subspace.

Problem 4. Express $v = (1, -2, 5)$ as a linear combination of $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, $v_3 = (2, -1, 1)$.

$a(1, 1, 1) + b(1, 2, 3) + c(2, -1, 1) = (1, -2, 5)$ gives $a + b + 2c = 1$, $a + 2b - c = -2$, $a + 3b + c = 5$.

$R2 - R1$: $b - 3c = -3$. $R3 - R1$: $2b - c = 4$. From these: $b = 3c - 3 \Rightarrow 2(3c - 3) - c = 4 \Rightarrow 5c = 10 \Rightarrow c = 2$, $b = 3$.

$a = 1 - b - 2c = 1 - 3 - 4 = -6$.



Answer: $v = -6v_1 + 3v_2 + 2v_3$.

Problem 5. Test whether the vectors $(1, -1, 2)$, $(2, 1, -1)$, $(3, 0, 1)$ are linearly independent. If dependent, find the relation.

Determinant of the matrix with these rows: $1(1 \cdot 1 - (-1) \cdot 0) - (-1)(2 \cdot 1 - (-1) \cdot 3) + 2(2 \cdot 0 - 1 \cdot 3) = 1 + 5 - 6 = 0$.

So the vectors are LD. Solve $a(1, -1, 2) + b(2, 1, -1) + c(3, 0, 1) = 0$: from the second component $-a + b = 0 \Rightarrow a = b$. From the first $a + 2b + 3c = 0 \Rightarrow 3a = -3c \Rightarrow c = -a$.

Take $a = 1$: $b = 1$, $c = -1$.

Answer: Linearly dependent: $v_1 + v_2 - v_3 = 0$ (i.e. $v_3 = v_1 + v_2$).

Problem 6. Show that $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of R^3 and find the coordinates of $(3, 2, 5)$.

Determinant of the matrix with rows $(1,0,0)$, $(1,1,0)$, $(1,1,1) = 1$ (triangular), which is non-zero, so the three vectors are LI. Three LI vectors in R^3 form a basis.

$a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1) = (3, 2, 5)$: third component $c = 5$; second $b + c = 2 \Rightarrow b = -3$; first $a + b + c = 3 \Rightarrow a = 1$.

Answer: It is a basis; coordinates of $(3, 2, 5)$ are $(1, -3, 5)$.

Problem 7. Find the dimension and a basis of the subspace of R^4 spanned by $(1, 2, 3, 0)$, $(2, 4, 3, 2)$, $(3, 2, 1, 3)$, $(6, 8, 7, 5)$.

Form the matrix with these rows and reduce. $R_2 - 2R_1 \rightarrow (0, 0, -3, 2)$; $R_3 - 3R_1 \rightarrow (0, -4, -8, 3)$; $R_4 - 6R_1 \rightarrow (0, -4, -11, 5)$.

Swap to get $(1, 2, 3, 0)$, $(0, -4, -8, 3)$, $(0, 0, -3, 2)$, $(0, -4, -11, 5)$. $R_4 - R_2 \rightarrow (0, 0, -3, 2)$. $R_4 - R_3 \rightarrow (0, 0, 0, 0)$.

Three non-zero rows remain.

Answer: Dimension 3; basis $\{(1, 2, 3, 0), (0, -4, -8, 3), (0, 0, -3, 2)\}$. (Also $v_4 = v_1 + v_2 + v_3$.)

Problem 8. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

$R_2 - 2R_1 \rightarrow (0, 0, 0)$; $R_3 - 3R_1 \rightarrow (0, 0, 0)$.

Answer: rank $A = 1$ (all rows are multiples of the first).

Problem 9. Find the rank of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$.

$R_2 - R_1 \rightarrow (0, 1, 2)$; $R_3 - R_1 \rightarrow (0, 2, 4)$. $R_3 - 2R_2 \rightarrow (0, 0, 0)$.

Echelon form has two non-zero rows. (Also $|A| = 0$ while the 2×2 minor $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$.)

Answer: rank $A = 2$.

Problem 10. Find the rank of $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ by reducing to echelon form.

$R_2 - 4R_1 \rightarrow (0, -7, 6, -11)$; $R_3 - 3R_1 \rightarrow (0, -7, 4, -7)$; $R_4 - R_1 \rightarrow (0, 0, 1, -2)$.



$$R_3 - R_2 \rightarrow (0, 0, -2, 4). R_3 + 2R_4 \rightarrow (0, 0, 0, 0).$$

Non-zero rows: $(1, 2, -1, 3), (0, -7, 6, -11), (0, 0, 1, -2)$.

Answer: rank A = 3 (and $|A| = 0$).

Problem 11. Reduce $A = [1 \ 2 \ 3; 2 \ 3 \ 4; 3 \ 4 \ 5]$ to normal form and find its rank and a basis of its null space.

$R_2 - 2R_1 \rightarrow (0, -1, -2); R_3 - 3R_1 \rightarrow (0, -2, -4); R_3 - 2R_2 \rightarrow (0, 0, 0). R_2 \rightarrow -R_2$ gives $(0, 1, 2). R_1 - 2R_2 \rightarrow (1, 0, -1).$

RREF = $[1 \ 0 \ -1; 0 \ 1 \ 2; 0 \ 0 \ 0]$. Column operations $C_3 + C_1 - 2C_2$ give $[1 \ 0 \ 0; 0 \ 1 \ 0; 0 \ 0 \ 0] = [I_2 \ 0; 0 \ 0]$.

Null space: $x - z = 0, y + 2z = 0 \Rightarrow (x, y, z) = z(1, -2, 1).$

Answer: rank 2; normal form $[I_2 \ 0; 0 \ 0]$; null space basis $\{(1, -2, 1)\}$; nullity 1 (and $2 + 1 = 3$ verifies rank-nullity).

Problem 12. Solve by Gauss elimination: $x + y + z = 6, x + 2y + 3z = 14, x + 4y + 9z = 36$.

Augmented matrix $[1 \ 1 \ 1 \mid 6; 1 \ 2 \ 3 \mid 14; 1 \ 4 \ 9 \mid 36]$.

$R_2 - R_1 \rightarrow [0 \ 1 \ 2 \mid 8]; R_3 - R_1 \rightarrow [0 \ 3 \ 8 \mid 30]. R_3 - 3R_2 \rightarrow [0 \ 0 \ 2 \mid 6].$

Back substitution: $2z = 6 \Rightarrow z = 3; y + 2z = 8 \Rightarrow y = 2; x + y + z = 6 \Rightarrow x = 1.$

Answer: $x = 1, y = 2, z = 3$.

Problem 13. Solve by Gauss elimination: $2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16$.

Take $R_1 \leftrightarrow R_3$ so that the pivot is 1: $[1 \ 4 \ 9 \mid 16; 3 \ 2 \ 3 \mid 18; 2 \ 1 \ 1 \mid 10].$

$R_2 - 3R_1 \rightarrow [0 \ -10 \ -24 \mid -30]; R_3 - 2R_1 \rightarrow [0 \ -7 \ -17 \mid -22].$

$R_2 \rightarrow R_2/(-2): [0 \ 5 \ 12 \mid 15]. R_3 \rightarrow 5R_3 + 7R_2: [0 \ 0 \ -85 + 84 \mid -110 + 105] = [0 \ 0 \ -1 \mid -5].$

$z = 5; 5y + 12(5) = 15 \Rightarrow y = -9; x + 4(-9) + 9(5) = 16 \Rightarrow x = 7.$

Answer: $x = 7, y = -9, z = 5$.

Problem 14. Solve $x + 2y + 3z = 1, 2x + y + 3z = 2, 5x + y + z = 3$ by Gauss elimination.

$R_2 - 2R_1 \rightarrow [0 \ -3 \ -3 \mid 0]; R_3 - 5R_1 \rightarrow [0 \ -9 \ -14 \mid -2].$

$R_3 - 3R_2 \rightarrow [0 \ 0 \ -5 \mid -2] \Rightarrow z = 2/5.$

$-3y - 3z = 0 \Rightarrow y = -z = -2/5. x = 1 - 2y - 3z = 1 + 4/5 - 6/5 = 3/5.$

Answer: $x = 3/5, y = -2/5, z = 2/5$.

Problem 15. Test the consistency of $x + 3y - 2z = 5, 2x - y + 4z = 3, x - 11y + 14z = -7$ and solve if consistent.

$[1 \ 3 \ -2 \mid 5; 2 \ -1 \ 4 \mid 3; 1 \ -11 \ 14 \mid -7]. R_2 - 2R_1 \rightarrow [0 \ -7 \ 8 \mid -7]; R_3 - R_1 \rightarrow [0 \ -14 \ 16 \mid -12].$

$R_3 - 2R_2 \rightarrow [0 \ 0 \ 0 \mid 2].$



rank $A = 2$ but rank $[A | B] = 3$.

Answer: Inconsistent; no solution.

Problem 16. Test the consistency of $x + y + z = 6$, $x + 2y + 3z = 14$, $2x + 3y + 4z = 20$ and solve.

$$R_2 - R_1 \rightarrow [0 \ 1 \ 2 \ | \ 8]; R_3 - 2R_1 \rightarrow [0 \ 1 \ 2 \ | \ 8]; R_3 - R_2 \rightarrow [0 \ 0 \ 0 \ | \ 0].$$

rank $A = \text{rank } [A | B] = 2 < 3$ unknowns $\Rightarrow$ infinitely many solutions with $3 - 2 = 1$ free parameter.

$$\text{Let } z = k. y = 8 - 2k, x = 6 - y - z = -2 + k.$$

Answer: Consistent; $x = k - 2$, $y = 8 - 2k$, $z = k$ for any real k .

Problem 17. For what values of λ and μ does the system $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (i) no solution, (ii) a unique solution, (iii) infinitely many solutions?

$$R_2 - R_1 \rightarrow [0 \ 1 \ 2 \ | \ 4]; R_3 - R_1 \rightarrow [0 \ 1 \ \lambda - 1 \ | \ \mu - 6]; R_3 - R_2 \rightarrow [0 \ 0 \ \lambda - 3 \ | \ \mu - 10].$$

(i) $\lambda = 3$, $\mu \neq 10$: rank $A = 2$, rank $[A|B] = 3 \Rightarrow$ no solution.

(ii) $\lambda \neq 3$: rank $A = \text{rank } [A|B] = 3 \Rightarrow$ unique solution (any μ).

(iii) $\lambda = 3$, $\mu = 10$: rank $A = \text{rank } [A|B] = 2 < 3 \Rightarrow$ infinitely many solutions.

Answer: No solution: $\lambda = 3$, $\mu \neq 10$. Unique: $\lambda \neq 3$. Infinite: $\lambda = 3$, $\mu = 10$.

Problem 18. Solve the homogeneous system $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$.

$$R_2 - 3R_1 \rightarrow (0, -2, -5); R_3 - 7R_1 \rightarrow (0, -4, -9). R_3 - 2R_2 \rightarrow (0, 0, 1).$$

rank $A = 3 =$ number of unknowns $\Rightarrow$ only the trivial solution.

Answer: $x = y = z = 0$.

Problem 19. Solve the homogeneous system $x + 2y - z = 0$, $2x + y + z = 0$, $x - y + 2z = 0$.

$R_2 - 2R_1 \rightarrow (0, -3, 3); R_3 - R_1 \rightarrow (0, -3, 3); R_3 - R_2 \rightarrow (0, 0, 0)$. rank $= 2 < 3$, so non-trivial solutions exist.

From $-3y + 3z = 0 \Rightarrow y = z$. From $x + 2y - z = 0 \Rightarrow x = -z$. Let $z = k$.

Answer: $x = -k$, $y = k$, $z = k$ (a one-parameter family; solution space of dimension 1).

Problem 20. For what value of k does the system $x + y + z = 0$, $2x + y - z = 0$, $3x + 2y + kz = 0$ have non-trivial solutions?

A homogeneous square system has non-trivial solutions iff $|A| = 0$.

$$|A| = 1(k + 2) - 1(2k + 3) + 1(4 - 3) = k + 2 - 2k - 3 + 1 = -k.$$

Answer: $k = 0$. (Then solution: from $x + y + z = 0$ and $2x + y - z = 0 \Rightarrow x = 2t$, $y = -3t$, $z = t$.)



Problem 21. Show that the vectors $(1, 2, 1)$, $(2, 1, 0)$, $(1, -1, 2)$ form a basis of $\mathbb{R}^3$.

$$|1 \ 2 \ 1; 2 \ 1 \ 0; 1 \ -1 \ 2| = 1(2 - 0) - 2(4 - 0) + 1(-2 - 1) = 2 - 8 - 3 = -9 \neq 0.$$

Answer: The vectors are LI; three LI vectors in $\mathbb{R}^3$ form a basis.

4.17 Practice problems with solutions: Module IV

Q1. Is $W = \{(x, y, z) : x = y = z\}$ a subspace of $\mathbb{R}^3$? Find a basis.

$$W = \{t(1, 1, 1)\}: \text{contains } 0, \text{ and } a \cdot t_1(1, 1, 1) + b \cdot t_2(1, 1, 1) = (at_1 + bt_2)(1, 1, 1) \in W.$$

Answer: Yes; basis $\{(1, 1, 1)\}$; dimension 1 (a line through the origin).

Q2. Is $W = \{(x, y) : xy \geq 0\}$ a subspace of $\mathbb{R}^2$?

$$(1, 0) \in W \text{ and } (0, -1) \in W, \text{ but their sum } (1, -1) \text{ has } xy = -1 < 0.$$

Answer: No (not closed under addition).

Q3. Are $(1, 2, 3)$, $(4, 5, 6)$, $(7, 8, 9)$ linearly independent?

$$|1 \ 2 \ 3; 4 \ 5 \ 6; 7 \ 8 \ 9| = 1(45 - 48) - 2(36 - 42) + 3(32 - 35) = -3 + 12 - 9 = 0 \Rightarrow \text{LD}.$$

$$\text{Relation: } (1, 2, 3) + (7, 8, 9) = (8, 10, 12) = 2(4, 5, 6).$$

Answer: LD; $v_1 - 2v_2 + v_3 = 0$.

Q4. Are $1 + x$, $x + x^2$, $1 + x^2$ linearly independent in P_2 ?

$$a(1 + x) + b(x + x^2) + c(1 + x^2) = 0 \Rightarrow \text{constant: } a + c = 0; x: a + b = 0; x^2: b + c = 0.$$

$$\text{Adding: } 2(a + b + c) = 0 \Rightarrow a + b + c = 0 \Rightarrow b = 0, c = 0, a = 0.$$

Answer: LI (they form a basis of P_2).

Q5. Find the dimension of the space of 3×3 symmetric matrices.

A symmetric matrix is fixed by its 3 diagonal entries and the 3 entries above the diagonal.

Answer: 6.

Q6. Find the dimension of the space of 3×3 skew-symmetric matrices.

Diagonal entries are 0; the 3 entries above the diagonal determine the rest.

Answer: 3.

Q7. Find the rank of $[2 \ 3 \ -1 \ -1; 1 \ -1 \ -2 \ -4; 3 \ 1 \ 3 \ -2; 6 \ 3 \ 0 \ -7]$.

$$R_1 \leftrightarrow R_2: (1, -1, -2, -4). R_2 - 2R_1 \rightarrow (0, 5, 3, 7); R_3 - 3R_1 \rightarrow (0, 4, 9, 10); R_4 - 6R_1 \rightarrow (0, 9, 12, 17).$$

$$R_4 - R_2 - R_3 \rightarrow (0, 0, 0, 0). R_2 \text{ and } R_3 \text{ are not proportional.}$$

Answer: rank = 3.

Q8. Find the rank of $[1 \ 2 \ 3; 4 \ 5 \ 6; 7 \ 8 \ 9]$.

$$R_2 - 4R_1 \rightarrow (0, -3, -6); R_3 - 7R_1 \rightarrow (0, -6, -12); R_3 - 2R_2 \rightarrow 0.$$



Answer: rank = 2.

Q9. Solve $2x - y + 3z = 9$, $x + y + z = 6$, $x - y + z = 2$.

Subtract the third from the second: $2y = 4 \Rightarrow y = 2$. Then $x + z = 4$ and $2x + 3z = 11 \Rightarrow z = 3$, $x = 1$.

Answer: $x = 1$, $y = 2$, $z = 3$.

Q10. Solve $10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$.

Adding all: $12(x + y + z) = 36 \Rightarrow x + y + z = 3$. Then $9x + 3 = 12 \Rightarrow x = 1$; similarly $y = z = 1$.

Answer: $x = y = z = 1$.

Q11. Test consistency and solve $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$.

$R_2 - R_1 \rightarrow (0, 1, 2 \mid 1)$; $R_3 - R_1 \rightarrow (0, 3, 8 \mid 3)$; $R_3 - 3R_2 \rightarrow (0, 0, 2 \mid 0) \Rightarrow z = 0$, $y = 1$, $x = 2$.

Answer: Consistent, unique solution $(2, 1, 0)$.

Q12. For what λ does $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ have a solution? Solve in each case.

$R_2 - R_1 \rightarrow (0, 1, 3 \mid \lambda - 1)$; $R_3 - R_1 \rightarrow (0, 3, 9 \mid \lambda^2 - 1)$; $R_3 - 3R_2 \rightarrow (0, 0, 0 \mid \lambda^2 - 3\lambda + 2)$.

Consistent iff $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1$ or 2 .

$\lambda = 1$: $y = -3z$, $x = 1 + 2z$. $\lambda = 2$: $y = 1 - 3z$, $x = 2z$.

Answer: $\lambda = 1$: $(1 + 2k, -3k, k)$; $\lambda = 2$: $(2k, 1 - 3k, k)$.

Q13. Solve $x + y + z = 0$, $x - y + 2z = 0$, $2x + 3z = 0$.

$R_2 - R_1 \rightarrow (0, -2, 1)$; $R_3 - 2R_1 \rightarrow (0, -2, 1) \Rightarrow \text{rank } 2$. $z = 2k \Rightarrow y = k$, $x = -3k$.

Answer: $(-3k, k, 2k)$.

Q14. Balance the chemical equation $C_3H_8 + O_2 \rightarrow CO_2 + H_2O$ using a linear system.

$a C_3H_8 + b O_2 \rightarrow c CO_2 + d H_2O$. Carbon: $3a = c$. Hydrogen: $8a = 2d$. Oxygen: $2b = 2c + d$.

Take $a = 1$: $c = 3$, $d = 4$, $b = (6 + 4)/2 = 5$.

Answer: $C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$.

4.18 MCQs: Module IV

1. Which of the following is a subspace of R^3 ?

- (a) $\{(x, y, z) : x + y + z = 0\}$ (b) $\{(x, y, z) : xyz = 0\}$ (c) $\{(x, y, z) : x \geq 0\}$ (d) $\{(x, y, z) : x + y + z = 1\}$

□ (a) — Plane through the origin.



- 2.** Every subspace must contain
(a) the vector $(1, 1, 1)$ (b) a basis of $\mathbb{R}^n$ (c) only positive vectors (d) the zero vector
□ (d) — Closure under scalar multiplication by 0.
- 3.** A line in $\mathbb{R}^2$ is a subspace if and only if it
(a) is horizontal (b) has positive slope (c) passes through the origin (d) is vertical
□ (c) — Must contain 0.
- 4.** The set of polynomials of degree exactly 2 is
(a) a subspace of P_1 (b) not a vector space (c) a vector space (d) of dimension 3
□ (b) — Not closed under addition.
- 5.** Any set of vectors containing the zero vector is
(a) a spanning set (b) linearly dependent (c) linearly independent (d) a basis
□ (b) — $1 \cdot 0 = 0$ is a non-trivial relation.
- 6.** Two vectors are linearly dependent iff
(a) one is a scalar multiple of the other (b) they are orthogonal (c) their sum is zero (d) they have equal length
□ (a) — Definition.
- 7.** The maximum number of LI vectors in $\mathbb{R}^4$ is
(a) 4 (b) 5 (c) infinite (d) 3
□ (a) — $\dim \mathbb{R}^4 = 4$.
- 8.** Any 4 vectors in $\mathbb{R}^3$ are
(a) LI (b) a basis (c) orthogonal (d) LD
□ (d) — More than $\dim$ vectors.
- 9.** The vectors $(1, 2)$ and $(2, 4)$ are
(a) LD (b) a basis of $\mathbb{R}^2$ (c) orthogonal (d) LI
□ (a) — $(2, 4) = 2(1, 2)$.
- 10.** The vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ are
(a) LD (b) LI but do not span $\mathbb{R}^3$ (c) LI and form a basis of $\mathbb{R}^3$ (d) span $\mathbb{R}^2$
□ (c) — Standard basis.
- 11.** Three vectors in $\mathbb{R}^3$ form a basis iff the determinant of the matrix formed by them is
(a) one (b) zero (c) non-zero (d) positive
□ (c) — Non-zero determinant $\Leftrightarrow$ LI.
- 12.** The dimension of $\mathbb{R}^n$ is
(a) $n + 1$ (b) n (c) n^2 (d) 1
□ (b) — Standard basis has n vectors.
- 13.** The dimension of P_3 (polynomials of degree ≤ 3) is
(a) 6 (b) 2 (c) 3 (d) 4
□ (d) — Basis $1, x, x^2, x^3$.
- 14.** The dimension of $M_{2 \times 3}$ is
(a) 2 (b) 3 (c) 6 (d) 5
□ (c) — 2×3 .
- 15.** The dimension of the space of all 3×3 symmetric matrices is
(a) 3 (b) 6 (c) 4 (d) 9
□ (b) — $n(n + 1)/2$.
- 16.** The dimension of the space of all $n \times n$ skew-symmetric matrices is
(a) n^2 (b) $n(n - 1)/2$ (c) $n(n + 1)/2$ (d) n
□ (b) — Diagonal entries are zero.



- 17.** The dimension of the plane $x + y + z = 0$ in $\mathbb{R}^3$ is
(a) 2 (b) 0 (c) 1 (d) 3
□ (a) — One equation removes one degree of freedom.
- 18.** The dimension of the zero space $\{0\}$ is
(a) 0 (b) 1 (c) ∞ (d) undefined
□ (a) — Its basis is empty.
- 19.** A basis of a vector space is a set that is
(a) LI only (b) LI and spanning (c) orthogonal (d) spanning only
□ (b) — Definition.
- 20.** The span of a set of vectors is always
(a) a subspace (b) LI (c) finite (d) a basis
□ (a) — Closed under linear combinations.
- 21.** The rank of a matrix equals the number of
(a) zero rows (b) non-zero rows in its echelon form (c) rows (d) columns
□ (b) — Practical definition.
- 22.** The rank of the 3×3 identity matrix is
(a) 9 (b) 0 (c) 1 (d) 3
□ (d) — $|I| \neq 0$.
- 23.** The rank of a zero matrix is
(a) its order (b) undefined (c) 0 (d) 1
□ (c) — No non-zero minor.
- 24.** The rank of $[1 \ 2; 2 \ 4]$ is
(a) 1 (b) 2 (c) 0 (d) 4
□ (a) — Rows proportional; $\det = 0$.
- 25.** The rank of $[1 \ 2 \ 3; 2 \ 4 \ 6; 3 \ 6 \ 9]$ is
(a) 3 (b) 2 (c) 0 (d) 1
□ (d) — All rows multiples of (1, 2, 3).
- 26.** The rank of a 3×5 matrix is at most
(a) 5 (b) 3 (c) 15 (d) 8
□ (b) — $\leq \min(m, n)$.
- 27.** If A is $n \times n$ and $|A| \neq 0$, then rank A equals
(a) n (b) $n - 1$ (c) 0 (d) 1
□ (a) — Non-singular.
- 28.** $\text{rank}(A^T)$ equals
(a) $\text{rank}(A) + 1$ (b) $\text{rank}(A)$ (c) $n - \text{rank } A$ (d) 0
□ (b) — Row rank = column rank.
- 29.** Which is NOT an elementary row operation?
(a) $R_i \rightarrow kR_i$, $k \neq 0$ (b) $R_i \rightarrow R_i \times R_j$ (c) $R_i \leftrightarrow R_j$ (d) $R_i \rightarrow R_i + kR_j$
□ (b) — Product of rows is not allowed.
- 30.** Elementary row operations do NOT change
(a) the rank (b) the entries (c) the trace (d) the determinant
□ (a) — Rank is invariant (determinant may change sign or scale).
- 31.** A system $AX = B$ is consistent iff
(a) $\text{rank } A = \text{rank } [A|B]$ (b) $B = 0$ (c) $|A| = 0$ (d) $\text{rank } A < \text{rank } [A|B]$
□ (a) — Rouché-Capelli theorem.



- 32.** If $\text{rank } A = \text{rank } [A|B] = \text{number of unknowns}$, the system has
(a) a unique solution (b) no solution (c) exactly two solutions (d) infinitely many solutions
□ (a) — Consistency theorem.
- 33.** If $\text{rank } A = \text{rank } [A|B] < \text{number of unknowns}$, the system has
(a) a unique solution (b) infinitely many solutions (c) no solution (d) only the trivial solution
□ (b) — $n - r$ free parameters.
- 34.** If $\text{rank } A \neq \text{rank } [A|B]$, the system is
(a) inconsistent (b) consistent (c) unique (d) homogeneous
□ (a) — No solution.
- 35.** A homogeneous system $AX = 0$ is always
(a) consistent (b) undefined (c) inconsistent (d) uniquely solvable with non-zero solution
□ (a) — $X = 0$ is a solution.
- 36.** A square homogeneous system $AX = 0$ has non-trivial solutions iff
(a) $A = I$ (b) $|A| \neq 0$ (c) $\text{rank } A = n$ (d) $|A| = 0$
□ (d) — $\text{rank} < n$.
- 37.** If a system of 3 equations in 3 unknowns has $\text{rank } A = 2$ and $\text{rank } [A|B] = 2$, the number of free parameters is
(a) 3 (b) 0 (c) 1 (d) 2
□ (c) — $n - r = 3 - 2$.
- 38.** Gauss elimination reduces the coefficient matrix to
(a) lower triangular form (b) upper triangular form (c) identity (d) diagonal form
□ (b) — Then back substitution.
- 39.** Gauss-Jordan method reduces the coefficient matrix to
(a) upper triangular (b) lower triangular (c) symmetric (d) diagonal/identity (RREF)
□ (d) — Zeros above and below pivots.
- 40.** Partial pivoting in Gauss elimination is used to
(a) find eigenvalues (b) increase the rank (c) make the matrix symmetric (d) reduce round-off error
□ (d) — Largest pivot improves stability.
- 41.** The solution of $x + y = 3$, $x - y = 1$ is
(a) (2, 1) (b) (1, 2) (c) (3, 0) (d) (0, 3)
□ (a) — Add: $2x = 4$.
- 42.** The system $x + y = 2$, $2x + 2y = 5$ has
(a) no solution (b) infinitely many solutions (c) unique solution (d) two solutions
□ (a) — Parallel lines.
- 43.** The system $x + y = 2$, $2x + 2y = 4$ has
(a) no solution (b) unique solution (c) infinitely many solutions (d) only (1, 1)
□ (c) — Same line.
- 44.** The rank-nullity theorem for an $m \times n$ matrix says $\text{rank} + \text{nullity}$ equals
(a) m (b) n (c) mn (d) $m + n$
□ (b) — Number of columns.
- 45.** If A is 4×6 with rank 4, the nullity of A is
(a) 2 (b) 6 (c) 4 (d) 0
□ (a) — $6 - 4$.



- 46.** The solution set of a homogeneous system $AX = 0$ is
(a) a subspace (b) always empty (c) a single point always (d) never a subspace
□ (a) — Null space.
- 47.** The number of solutions of a consistent system with more unknowns than equations is
(a) one (b) zero (c) two (d) infinite
□ (d) — $\text{rank} \leq m < n$.
- 48.** A matrix is in row echelon form if
(a) all entries are 1 (b) it is symmetric (c) it is diagonal (d) zeros below each pivot and zero rows at the bottom
□ (d) — Definition of REF.
- 49.** The coordinates of $(3, 4)$ relative to the basis $\{(1, 0), (1, 1)\}$ are
(a) $(-1, 4)$ (b) $(3, 4)$ (c) $(1, 3)$ (d) $(4, -1)$
□ (a) — $a + b = 3, b = 4 \Rightarrow a = -1$.
- 50.** If $v_3 = 2v_1 - v_2$, then $\{v_1, v_2, v_3\}$ is
(a) a basis of $\mathbb{R}^3$ (b) LI (c) LD (d) orthogonal
□ (c) — Non-trivial relation $2v_1 - v_2 - v_3 = 0$.

Module V: Linear Algebra II (Inverse, Special Matrices, Eigenvalues)

This module continues with square matrices: how to invert them by row operations, the special classes of matrices that appear in engineering (symmetric, skew-symmetric, orthogonal), and the eigenvalue problem $AX = \lambda X$, one of the most useful ideas in all of applied mathematics (vibrations, stability, principal stresses, Google PageRank, quantum mechanics).

5.1 Inverse of a matrix

A square matrix A is **invertible (non-singular)** if there is a matrix B with $AB = BA = I$. Then B is unique and is written A^{-1} .

$$A^{-1} \text{ exists} \Leftrightarrow |A| \neq 0 \Leftrightarrow \text{rank } A = n$$

$$A^{-1} = \text{adj}(A)/|A| \text{ (adjoint method)}$$

$$(AB)^{-1} = B^{-1}A^{-1} \quad (A^T)^{-1} = (A^{-1})^T \quad (A^{-1})^{-1} = A \quad |A^{-1}| = 1/|A| \quad (kA)^{-1} = (1/k)A^{-1}$$

$$\text{For } 2 \times 2: [a \ b; c \ d]^{-1} = (1/(ad - bc)) [d \ -b; -c \ a]$$

The adjoint method needs n^2 cofactors and is slow for large n . The Gauss-Jordan method below is systematic and efficient.

5.2 Gauss-Jordan method for the inverse

Principle. Every elementary row operation is the same as multiplying on the left by an elementary matrix E . If a sequence of row operations $E_k \dots E_2 E_1$ turns A into I , then $E_k \dots E_1 = A^{-1}$. Applying the **same** operations to I therefore produces A^{-1} .

Method.

- Write the augmented matrix $[A \mid I]$ (A on the left, identity of the same order on the right).
- Apply elementary **row** operations only, to reduce the left block to I . Work column by column: make the pivot 1, then make every other entry in that column 0 (both below and above the pivot).
- When the left block becomes I , the right block is A^{-1} : $[A \mid I] \rightarrow [I \mid A^{-1}]$.
- If a zero row appears in the left block, A is singular and has no inverse.

Tip. Always verify by checking $AA^{-1} = I$ for at least one row and column.

5.3 Symmetric, skew-symmetric and orthogonal matrices

Type	Definition	Key properties	
Symmetric	$A^T = A$, i.e. $a_{ij} = a_{ji}$	Real eigenvalues; eigenvectors for distinct eigenvalues are orthogonal; can be diagonalised by an orthogonal matrix	
Skew-symmetric	$A^T = -A$, i.e. $a_{ij} = -a_{ji}$	Diagonal entries are 0; eigenvalues are 0 or purely imaginary; determinant of odd order is 0	

Type	Definition	Key properties	
Orthogonal	$A^T A = A A^T = I$, i.e. $A^{-1} = A^T$		A

Important results.

Every square matrix $A = P + Q$, where $P = (A + A^T)/2$ is symmetric and $Q = (A - A^T)/2$ is skew-symmetric (unique decomposition)

- For any matrix A , AA^T and $A^T A$ are symmetric.
- If A and B are symmetric, AB is symmetric iff $AB = BA$.
- The diagonal of a skew-symmetric matrix is zero because $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$.
- For a skew-symmetric matrix of odd order n : $|A| = |A^T| = |-A| = (-1)^n |A| = -|A| \Rightarrow |A| = 0$.
- If A is orthogonal: $|A|^2 = |A^T| |A| = |I| = 1 \Rightarrow |A| = \pm 1$. The product of two orthogonal matrices is orthogonal; the inverse of an orthogonal matrix is orthogonal.
- Rotation matrix** $[\cos \theta \ -\sin \theta; \sin \theta \ \cos \theta]$ is orthogonal with determinant $+1$. Reflection matrices are orthogonal with determinant -1 .
- Complex analogues: **Hermitian** ($A^* = A$, conjugate transpose), **skew-Hermitian** ($A^* = -A$), **unitary** ($A^* A = I$).

5.4 Eigenvalues and eigenvectors

Definition. Let A be an $n \times n$ matrix. A number λ is an **eigenvalue** (characteristic root, latent root) of A if there is a **non-zero** vector X such that

$$AX = \lambda X$$

X is an **eigenvector** belonging to λ . Geometrically, A only stretches (or shrinks, or reverses) X ; it does not change its direction.

How to find them. $AX = \lambda X \Leftrightarrow (A - \lambda I)X = 0$. A non-zero solution exists iff

Characteristic equation: $|A - \lambda I| = 0$

$A - \lambda I$	is a polynomial of degree n in λ , called the characteristic polynomial . Its n roots (counted with multiplicity) are the eigenvalues .
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Short cuts for writing the characteristic equation.

2×2 : $\lambda^2 - (\text{tr } A) \lambda + |A| = 0$
 3×3 : $\lambda^3 - (\text{tr } A) \lambda^2 + (S_2) \lambda - |A| = 0$, where S_2 = sum of the principal minors of order 2 (minors of the diagonal elements)

Steps to find eigenvectors. For each eigenvalue λ_i , solve $(A - \lambda_i I)X = 0$ by row reduction. The non-zero solutions (with free parameter) are the eigenvectors. For a 3×3 matrix with $\text{rank}(A - \lambda I) = 2$, use the cross-multiplication rule on any two independent rows: if the rows are (a_1, b_1, c_1) and (a_2, b_2, c_2) , then $x : y : z = (b_1 c_2 - b_2 c_1) : (c_1 a_2 - c_2 a_1) : (a_1 b_2 - a_2 b_1)$.

Algebraic and geometric multiplicity. The algebraic multiplicity (AM) of λ is its multiplicity as a root of the characteristic equation. The geometric multiplicity (GM) is the number of LI eigenvectors for $\lambda = n - \text{rank}(A - \lambda I)$. Always $1 \leq \text{GM} \leq \text{AM}$.

5.5 Properties of eigenvalues (very important for MCQs)

- (1) Sum of eigenvalues = trace of $A = a_{11} + a_{22} + \dots + a_{nn}$
 (2) Product of eigenvalues = $|A|$ (so A is singular $\Leftrightarrow 0$ is an eigenvalue)

- (3) A and A^T have the same eigenvalues.
- (4) Eigenvalues of a triangular or diagonal matrix are its diagonal entries.
- (5) If λ is an eigenvalue of A , then: $k\lambda$ is an eigenvalue of kA ; λ^m of A^m ; $1/\lambda$ of A^{-1} ($\lambda \neq 0$); $|A|/\lambda$ of $\text{adj } A$; $\lambda - k$ of $A - kI$; and $p(\lambda)$ of the polynomial $p(A)$. The eigenvectors stay the same.
- (6) Eigenvalues of a real symmetric matrix are real; of a skew-symmetric matrix are zero or purely imaginary; of an orthogonal matrix have $|\lambda| = 1$ (if λ is an eigenvalue, so is $1/\lambda$).
- (7) Eigenvectors corresponding to distinct eigenvalues are linearly independent. For a symmetric matrix they are also orthogonal.
- (8) An eigenvector cannot belong to two different eigenvalues; but one eigenvalue has infinitely many eigenvectors (all non-zero multiples, and combinations within its eigenspace).
- (9) Similar matrices ($B = P^{-1}AP$) have the same eigenvalues.
- (10) If all eigenvalues are distinct, A is diagonalisable: $P^{-1}AP = D$, where the columns of P are the eigenvectors and $D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Then $A^m = PD^mP^{-1}$.

5.6 Cayley-Hamilton theorem (statement only)

Statement. Every square matrix satisfies its own characteristic equation. If $|A - \lambda I| = (-1)^n(\lambda^n + c_1\lambda^{n-1} + \dots + c_{n-1}\lambda + c_n) = 0$, then

$$A^n + c_1A^{n-1} + \dots + c_{n-1}A + c_nI = O$$

Uses.

- **Finding A^{-1} :** multiply the equation by A^{-1} (possible when $|A| \neq 0$, i.e. $c_n \neq 0$): $A^{-1} = -(1/c_n)(A^{n-1} + c_1A^{n-2} + \dots + c_{n-1}I)$.
- **Finding high powers** such as A^4, A^8 or A^{100} in terms of lower powers (reduce using the equation, or divide a polynomial by the characteristic polynomial and keep the remainder).
- **Evaluating matrix polynomials** such as $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$.

For a 2×2 matrix: $A^2 - (\text{tr } A)A + |A|I = O$.

For a 3×3 matrix: $A^3 - (\text{tr } A)A^2 + S_2A - |A|I = O$.

Note: Common mistake: writing the scalar term as $|A|$ instead of $|A|I$. Every term of the Cayley-Hamilton equation must be a matrix.

5.7 Proofs of key eigenvalue properties (exam derivations)

(1) Sum of eigenvalues = trace, product = determinant. The characteristic polynomial is $|A - \lambda I| = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$. Comparing the coefficient of

λ^{n-1} on both sides gives $\lambda_1 + \dots + \lambda_n = a_{11} + \dots + a_{nn}$. Putting $\lambda = 0$ gives $|A| = \lambda_1 \lambda_2 \dots \lambda_n$.

(2) Eigenvalues of A^{-1} are $1/\lambda$. From $AX = \lambda X$ ($\lambda \neq 0$ since A is invertible), multiply by A^{-1} : $X = \lambda A^{-1}X$, so $A^{-1}X = (1/\lambda)X$.

(3) Eigenvalues of A^m are λ^m . $A^2X = A(\lambda X) = \lambda AX = \lambda^2X$; repeat.

(4) A and A^T have the same eigenvalues. $|A^T - \lambda I| = |(A - \lambda I)^T| = |A - \lambda I|$.

(5) Eigenvalues of a real symmetric matrix are real. Let $AX = \lambda X$ with X possibly complex. Take the conjugate transpose: $X^T A = \bar{\lambda} X^T$ (since A is real and symmetric). Multiply the first by X^T on the left and the second by X on the right: $X^T A X = \lambda X^T X$ and $X^T A X = \bar{\lambda} X^T X$. Since $X^T X = \sum |x_i|^2 > 0$, $\lambda = \bar{\lambda}$, i.e. λ is real.

(6) Eigenvectors of a symmetric matrix for distinct eigenvalues are orthogonal.

Let $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$. Then $\lambda_1 X_2^T X_1 = X_2^T A X_1 = (A X_2)^T X_1 = \lambda_2 X_2^T X_1$. So $(\lambda_1 - \lambda_2) X_2^T X_1 = 0$; since $\lambda_1 \neq \lambda_2$, $X_2^T X_1 = 0$.

(7) Eigenvalues of an orthogonal matrix have modulus 1. If $AX = \lambda X$, then $|AX|^2 = X^T A^T A X = X^T X = |X|^2$. Also $|AX|^2 = |\lambda|^2 |X|^2$. So $|\lambda| = 1$.

5.8 Diagonalisation and powers of a matrix

If an $n \times n$ matrix A has n linearly independent eigenvectors $X_1, \dots, X_n$ (always true when the eigenvalues are distinct), put them as columns of P (the **modal matrix**). Then

$$P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \quad A = PDP^{-1} \quad A^m = PD^mP^{-1}$$

Why. $AP = [AX_1 \dots AX_n] = [\lambda_1 X_1 \dots \lambda_n X_n] = PD$.

Orthogonal diagonalisation. For a real symmetric matrix, normalise the eigenvectors (divide each by its length). Then P is orthogonal, $P^{-1} = P^T$, and $P^T A P = D$. This is used to reduce a quadratic form $X^T A X$ to the sum of squares $\lambda_1 y_1^2 + \dots + \lambda_n y_n^2$ (principal axes).

Example. $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ has eigenvalues 0, 3, 15 with orthogonal eigenvectors (1, 2, 2), (2, 1, -2), (2, -2, 1), each of length 3. So $P = (1/3)[1 \ 2 \ 2; 2 \ 1 \ -2; 2 \ -2 \ 1]$ is orthogonal and $P^T A P = \text{diag}(0, 3, 15)$.

When is a matrix not diagonalisable? When some eigenvalue has GM < AM.

Example: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has $\lambda = 1$ twice but only one independent eigenvector (1, 0).

5.9 Quadratic forms (link to the Hessian in Module III)

A quadratic form in x, y, z is $Q = X^T A X$ with A symmetric. For example, $Q = 2x^2 + 2y^2 + 2z^2 + 2xy + 2yz + 2zx$ has $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

The **nature** of Q is decided by the eigenvalues of A :

Eigenvalues of A	Nature of Q
All positive	Positive definite ($Q > 0$ for $X \neq 0$)
All negative	Negative definite
All ≥ 0 , at least one zero	Positive semi-definite
Some positive, some negative	Indefinite

For the example, the eigenvalues are 1, 1, 4, all positive, so Q is positive definite. The Hessian test for maxima and minima in Module III is exactly this classification applied to the second-order part of Taylor's formula.

5.10 Engineering applications of eigenvalues

- **Vibrations:** for a system of masses and springs, the equations of motion are $M\ddot{X} + KX = 0$. The natural frequencies ω satisfy $|K - \omega^2 M| = 0$, an eigenvalue problem. The eigenvectors are the mode shapes.
- **Stability of systems:** a system $\dot{X} = AX$ is stable when every eigenvalue of A has a negative real part (control engineering).
- **Principal stresses:** the stress tensor is symmetric; its eigenvalues are the principal stresses and its eigenvectors the principal directions.
- **Markov chains and PageRank:** the steady state is the eigenvector for eigenvalue 1.
- **Quantum mechanics:** observable quantities are eigenvalues of operators (see Physics, Module IV).
- **Data science:** principal component analysis uses eigenvectors of the covariance matrix.

5.11 Common mistakes to avoid in Module V

- In Gauss-Jordan, mixing row and column operations. Use row operations only.
- Writing an eigenvector as the zero vector. Eigenvectors must be non-zero.
- Forgetting that the eigenvector is determined only up to a non-zero multiple; (1, 2, 2) and (-2, -4, -4) are the same answer.
- In Cayley-Hamilton, writing a scalar (like 6) instead of $6I$.
- Assuming $AB = BA$ when simplifying matrix expressions.

5.12 Solved problems: Module V

Problem 1. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ by the Gauss-Jordan method.

$$[A | I] = [1 \ 2 \ 3 \ | \ 1 \ 0 \ 0; 0 \ 1 \ 4 \ | \ 0 \ 1 \ 0; 5 \ 6 \ 0 \ | \ 0 \ 0 \ 1].$$

$$R_3 \rightarrow R_3 - 5R_1: [0 \ -4 \ -15 \ | \ -5 \ 0 \ 1].$$

$$R_3 \rightarrow R_3 + 4R_2: [0 \ 0 \ 1 \ | \ -5 \ 4 \ 1]. \text{ (Pivot is already 1.)}$$

$$R_2 \rightarrow R_2 - 4R_3: [0 \ 1 \ 0 \ | \ 20 \ -15 \ -4]. \quad R_1 \rightarrow R_1 - 3R_3: [1 \ 2 \ 0 \ | \ 16 \ -12 \ -3].$$

$$R_1 \rightarrow R_1 - 2R_2: [1 \ 0 \ 0 \ | \ -24 \ 18 \ 5].$$

$$\text{Check row 1 of } A \text{ times column 1 of } A^{-1}: 1(-24) + 2(20) + 3(-5) = 1 \quad \square.$$

Answer: $A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$.

Problem 2. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ by Gauss-Jordan.

$$[2 \ 0 \ 1 \ | \ 1 \ 0 \ 0; 0 \ 2 \ 0 \ | \ 0 \ 1 \ 0; 1 \ 0 \ 2 \ | \ 0 \ 0 \ 1]. \quad R_1 \leftrightarrow R_3: [1 \ 0 \ 2 \ | \ 0 \ 0 \ 1; 0 \ 2 \ 0 \ | \ 0 \ 1 \ 0; 2 \ 0 \ 1 \ | \ 1 \ 0 \ 0].$$

$$R_3 \rightarrow R_3 - 2R_1: [0 \ 0 \ -3 \ | \ 1 \ 0 \ -2]. \quad R_2 \rightarrow R_2/2: [0 \ 1 \ 0 \ | \ 0 \ 1/2 \ 0]. \quad R_3 \rightarrow R_3/(-3): [0 \ 0 \ 1 \ | \ -1/3 \ 0 \ 2/3].$$

$$R_1 \rightarrow R_1 - 2R_3: [1 \ 0 \ 0 \ | \ 2/3 \ 0 \ -1/3].$$

Answer: $A^{-1} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{bmatrix}$ ($|A| = 6$).

Problem 3. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ by Gauss-Jordan.

$$R_2 - R_1: [0 \ 2 \ -6 \ | \ -1 \ 1 \ 0]; \quad R_3 + 2R_1: [0 \ -2 \ 2 \ | \ 2 \ 0 \ 1].$$



$R_3 + R_2: [0 \ 0 \ -4 \ | \ 1 \ 1 \ 1] \Rightarrow R_3/(-4): [0 \ 0 \ 1 \ | \ -1/4 \ -1/4 \ -1/4].$

$R_2 + 6R_3: [0 \ 2 \ 0 \ | \ -5/2 \ -1/2 \ -3/2] \Rightarrow R_2/2: [0 \ 1 \ 0 \ | \ -5/4 \ -1/4 \ -3/4].$

$R_1 - 3R_3 - R_2: [1 \ 0 \ 0 \ | \ 1 + 3/4 + 5/4, 0 + 3/4 + 1/4, 0 + 3/4 + 3/4] = [1 \ 0 \ 0 \ | \ 3 \ 1 \ 3/2].$

Answer: $A^{-1} = [3 \ 1 \ 3/2; -5/4 \ -1/4 \ -3/4; -1/4 \ -1/4 \ -1/4].$

Problem 4. Show that $A = [1 \ 2 \ 3; 2 \ 4 \ 6; 1 \ 1 \ 1]$ has no inverse using the Gauss-Jordan method.

$R_2 - 2R_1$ gives $[0 \ 0 \ 0 \ | \ -2 \ 1 \ 0]$: a zero row appears in the left block.

Answer: A is singular ($|A| = 0$), so A^{-1} does not exist.

Problem 5. Express $A = [1 \ 2 \ 4; 6 \ 8 \ 1; 3 \ 5 \ 7]$ as the sum of a symmetric and a skew-symmetric matrix.

$A^T = [1 \ 6 \ 3; 2 \ 8 \ 5; 4 \ 1 \ 7].$

$P = (A + A^T)/2 = (1/2)[2 \ 8 \ 7; 8 \ 16 \ 6; 7 \ 6 \ 14] = [1 \ 4 \ 7/2; 4 \ 8 \ 3; 7/2 \ 3 \ 7]$ (symmetric).

$Q = (A - A^T)/2 = (1/2)[0 \ -4 \ 1; 4 \ 0 \ -4; -1 \ 4 \ 0] = [0 \ -2 \ 1/2; 2 \ 0 \ -2; -1/2 \ 2 \ 0]$ (skew-symmetric).

Answer: $A = P + Q$ with P and Q as above (check: $P + Q = A$).

Problem 6. Show that $A = (1/3)[1 \ 2 \ 2; 2 \ 1 \ -2; -2 \ 2 \ -1]$ is orthogonal and hence find A^{-1} .

Rows of $3A: r_1 = (1, 2, 2), r_2 = (2, 1, -2), r_3 = (-2, 2, -1)$. Each has length $\sqrt{9} = 3$, so rows of A have length 1.

$r_1 \cdot r_2 = 2 + 2 - 4 = 0; r_1 \cdot r_3 = -2 + 4 - 2 = 0; r_2 \cdot r_3 = -4 + 2 + 2 = 0$. Rows are orthonormal $\Rightarrow AA^T = I$.

Answer: A is orthogonal; $A^{-1} = A^T = (1/3)[1 \ 2 \ -2; 2 \ 1 \ 2; 2 \ -2 \ -1]$.

Problem 7. Find a, b, c if $A = [0 \ 2b \ c; a \ b \ -c; a \ -b \ c]$ is orthogonal.

Columns must be orthonormal. Column 1 $\cdot$ column 2: $0 + ab - ab = 0$ $\square$ always.

Column 2 $\cdot$ column 3: $2bc - bc - bc = 0$ $\square$.

Column 1 length: $2a^2 = 1 \Rightarrow a = \pm 1/\sqrt{2}$. Column 2: $4b^2 + b^2 + b^2 = 6b^2 = 1 \Rightarrow b = \pm 1/\sqrt{6}$. Column 3: $3c^2 = 1 \Rightarrow c = \pm 1/\sqrt{3}$.

Answer: $a = \pm 1/\sqrt{2}, b = \pm 1/\sqrt{6}, c = \pm 1/\sqrt{3}$.

Problem 8. Show that the rotation matrix $R = [\cos \theta \ -\sin \theta; \sin \theta \ \cos \theta]$ is orthogonal and find its eigenvalues.

$R^T R = [\cos \theta \ \sin \theta; -\sin \theta \ \cos \theta][\cos \theta \ -\sin \theta; \sin \theta \ \cos \theta] = [\cos^2 \theta + \sin^2 \theta, 0; 0, \sin^2 \theta + \cos^2 \theta] = I$.

Characteristic equation: $\lambda^2 - 2\cos \theta \lambda + 1 = 0 \Rightarrow \lambda = \cos \theta \pm i \sin \theta = e^{\pm i\theta}$.

Answer: R is orthogonal ($|R| = 1$); eigenvalues $e^{\pm i\theta}$, both of modulus 1.

Problem 9. Find the eigenvalues and eigenvectors of $A = [4 \ 1; 2 \ 3]$.



$$\lambda^2 - (\text{tr } A)\lambda + |A| = \lambda^2 - 7\lambda + 10 = 0 \Rightarrow \lambda = 2, 5.$$

$$\lambda = 2: (A - 2I)X = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} X = 0 \Rightarrow 2x + y = 0 \Rightarrow X = k(1, -2).$$

$$\lambda = 5: (A - 5I)X = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} X = 0 \Rightarrow x = y \Rightarrow X = k(1, 1).$$

Answer: $\lambda = 2$ with $X = (1, -2)^T$; $\lambda = 5$ with $X = (1, 1)^T$. (Check: sum 7 = trace, product 10 = $|A|$.)

Problem 10. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

$$\text{tr } A = 18. S_2 = |7 \ -4; \ -4 \ 3| + |8 \ 2; \ 2 \ 3| + |8 \ -6; \ -6 \ 7| = (21 - 16) + (24 - 4) + (56 - 36) = 5 + 20 + 20 = 45. |A| = 0.$$

$$\text{Characteristic equation: } \lambda^3 - 18\lambda^2 + 45\lambda = 0 \Rightarrow \lambda(\lambda - 3)(\lambda - 15) = 0 \Rightarrow \lambda = 0, 3, 15.$$

$$\lambda = 0: \text{rows } (8, -6, 2), (-6, 7, -4). \text{ Cross-multiplication: } x : y : z = (24 - 14) : (-12 + 32) : (56 - 36) = 10 : 20 : 20 = 1 : 2 : 2.$$

$$\lambda = 3: A - 3I \text{ rows } (5, -6, 2), (-6, 4, -4): x : y : z = (24 - 8) : (-12 + 20) : (20 - 36) = 16 : 8 : -16 = 2 : 1 : -2.$$

$$\lambda = 15: A - 15I \text{ rows } (-7, -6, 2), (-6, -8, -4): x : y : z = (24 + 16) : (-12 - 28) : (56 - 36) = 40 : -40 : 20 = 2 : -2 : 1.$$

Answer: $\lambda = 0, 3, 15$ with eigenvectors $(1, 2, 2)$, $(2, 1, -2)$, $(2, -2, 1)$. These are mutually orthogonal because A is symmetric.

Problem 11. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

$$\text{Expanding along row 2: } |A - \lambda I| = (2 - \lambda)[(2 - \lambda)^2 - 1] = (2 - \lambda)(1 - \lambda)(3 - \lambda) \Rightarrow \lambda = 1, 2, 3.$$

$$\lambda = 1: \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} X = 0 \Rightarrow x + z = 0, y = 0 \Rightarrow X = (1, 0, -1).$$

$$\lambda = 2: \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} X = 0 \Rightarrow x = z = 0 \Rightarrow X = (0, 1, 0).$$

$$\lambda = 3: \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} X = 0 \Rightarrow x = z, y = 0 \Rightarrow X = (1, 0, 1).$$

Answer: $\lambda = 1, 2, 3$; $X = (1, 0, -1)$, $(0, 1, 0)$, $(1, 0, 1)$.

Problem 12. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ (repeated eigenvalue).

$$\text{tr } A = 6, S_2 = 3 + 3 + 3 = 9, |A| = 4. \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \Rightarrow (\lambda - 1)^2(\lambda - 4) = 0 \Rightarrow \lambda = 1, 1, 4.$$

$$\lambda = 4: A - 4I \text{ rows } (-2, 1, 1), (1, -2, 1) \Rightarrow x : y : z = (1 + 2) : (1 + 2) : (4 - 1) = 1 : 1 : 1.$$

$$\lambda = 1: A - I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{rank } 1 \Rightarrow x + y + z = 0 \text{ with two free variables. GM} = 3 - 1 = 2.$$

$$\text{Two LI eigenvectors: } (1, -1, 0) \text{ and } (1, 0, -1).$$

Answer: $\lambda = 4$: $(1, 1, 1)$; $\lambda = 1$ (AM = GM = 2): $(1, -1, 0)$, $(1, 0, -1)$.

Problem 13. If the eigenvalues of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ are 2, 2, 3, find the eigenvalues of A^{-1} , A^2 , $3A - 2I$ and $\text{adj } A$.

$$|A| = \text{product} = 12. \text{tr } A = 3 - 3 + 7 = 7 = 2 + 2 + 3 \square.$$



A^{-1} : $1/2, 1/2, 1/3$. A^2 : $4, 4, 9$. $3A - 2I$: $3\lambda - 2 = 4, 4, 7$. $\text{adj } A$: $|A|/\lambda = 6, 6, 4$.

Answer: A^{-1} : $1/2, 1/2, 1/3$; A^2 : $4, 4, 9$; $3A - 2I$: $4, 4, 7$; $\text{adj } A$: $6, 6, 4$.

Problem 14. The product of two eigenvalues of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigenvalue and the sum of all eigenvalues.

$$|A| = 6(9 - 1) + 2(-6 + 2) + 2(2 - 6) = 48 - 8 - 8 = 32.$$

$$\text{Third eigenvalue} = |A|/16 = 2. \text{ Sum of eigenvalues} = \text{tr } A = 6 + 3 + 3 = 12.$$

Answer: Third eigenvalue 2; sum 12 (the eigenvalues are 2, 2, 8).

Problem 15. Verify the Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and find A^{-1} and A^4 .

$$\text{Characteristic equation: } \lambda^2 - 2\lambda - 3 = 0 \text{ (tr } A = 2, |A| = 1 - 4 = -3).$$

$$A^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}. A^2 - 2A - 3I = \begin{bmatrix} 5 - 2 - 3 & 4 - 4 - 0 \\ 4 - 4 - 0 & 5 - 2 - 3 \end{bmatrix} = O.$$

$$A^{-1}: \text{from } A^2 - 2A - 3I = O, \text{ multiply by } A^{-1}: A - 2I - 3A^{-1} = O \Rightarrow A^{-1} = (A - 2I)/3 = (1/3)\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

$$A^4: A^2 = 2A + 3I \Rightarrow A^3 = 2A^2 + 3A = 2(2A + 3I) + 3A = 7A + 6I \Rightarrow A^4 = 7A^2 + 6A = 7(2A + 3I) + 6A = 20A + 21I = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix}.$$

Answer: Verified; $A^{-1} = (1/3)\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$; $A^4 = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix}$.

Problem 16. Verify the Cayley-Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & 2 \end{bmatrix}$ and hence find A^{-1} .

$$\text{tr } A = 6, S_2 = (4 - 1) + (4 - 1) + (4 - 1) = 9, |A| = 2(4 - 1) + 1(-2 + 1) + 1(1 - 2) = 6 - 1 - 1 = 4.$$

$$\text{Characteristic equation: } \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}; A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}.$$

$$A^3 - 6A^2 + 9A - 4I: (1,1) \text{ entry } 22 - 36 + 18 - 4 = 0; (1,2) \text{ entry } -21 + 30 - 9 - 0 = 0; \text{ similarly all entries are } 0.$$

$$\text{Multiply by } A^{-1}: A^2 - 6A + 9I - 4A^{-1} = O \Rightarrow A^{-1} = (A^2 - 6A + 9I)/4.$$

$$A^2 - 6A + 9I = \begin{bmatrix} 6 - 12 + 9 & -5 + 6 & 5 - 6 \\ -5 + 6 & 6 - 12 + 9 & -5 + 6 \\ 5 - 6 & -5 + 6 & 6 - 12 + 9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

Answer: $A^{-1} = (1/4)\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Problem 17. Using Cayley-Hamilton, find A^8 if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

$$\text{tr } A = 0, |A| = -1 - 4 = -5 \Rightarrow \lambda^2 - 5 = 0 \Rightarrow A^2 = 5I.$$

$$A^8 = (A^2)^4 = (5I)^4 = 625 I.$$

Answer: $A^8 = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$.

Problem 18. Diagonalise $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$.

$$\text{From the earlier problem: } \lambda = 2, X_1 = (1, -2); \lambda = 5, X_2 = (1, 1).$$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, |P| = 3, P^{-1} = (1/3)\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

$$P^{-1}AP = D = \text{diag}(2, 5).$$



Answer: $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$.

5.13 Practice problems with solutions: Module V

Q1. Find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by Gauss-Jordan.

$[1 \ 2 \mid 1 \ 0; 3 \ 4 \mid 0 \ 1]$. $R_2 - 3R_1: [0 \ -2 \mid -3 \ 1]$. $R_2/(-2): [0 \ 1 \mid 3/2 \ -1/2]$. $R_1 - 2R_2: [1 \ 0 \mid -2 \ 1]$.

Answer: $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$.

Q2. Find the inverse of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan.

$R_1 \leftrightarrow R_2: [1 \ 2 \ 1 \mid 0 \ 1 \ 0; 2 \ 1 \ 1 \mid 1 \ 0 \ 0; 1 \ 1 \ 2 \mid 0 \ 0 \ 1]$. $R_2 - 2R_1: [0 \ -3 \ -1 \mid 1 \ -2 \ 0]$. $R_3 - R_1: [0 \ -1 \ 1 \mid 0 \ -1 \ 1]$.

$R_2 \leftrightarrow R_3$ and multiply by $-1: [0 \ 1 \ -1 \mid 0 \ 1 \ -1]$. R_3 (old R_2) $+ 3R_2: [0 \ 0 \ -4 \mid 1 \ 1 \ -3] \Rightarrow R_3/(-4): [0 \ 0 \ 1 \mid -1/4 \ -1/4 \ 3/4]$.

$R_2 + R_3: [0 \ 1 \ 0 \mid -1/4 \ 3/4 \ -1/4]$. $R_1 - 2R_2 - R_3: [1 \ 0 \ 0 \mid 0 + 1/2 + 1/4, 1 - 3/2 + 1/4, 0 + 1/2 - 3/4] = [1 \ 0 \ 0 \mid 3/4 \ -1/4 \ -1/4]$.

Answer: $A^{-1} = (1/4)[3 \ -1 \ -1; -1 \ 3 \ -1; -1 \ -1 \ 3]$.

Q3. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Upper triangular. $R_2 - R_3: [0 \ 1 \ 0 \mid 0 \ 1 \ -1]$. $R_1 - R_2 - R_3: [1 \ 0 \ 0 \mid 1 \ -1 \ 0]$.

Answer: $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Q4. Show that $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ is orthogonal.

$AA^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & 0 \\ 0 & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = I$.

Answer: Orthogonal ($|A| = 1$, a rotation).

Q5. Is $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ skew-symmetric? Find $|A|$.

$A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} = -A \Rightarrow$ skew-symmetric.

$|A| = 0(0 + 9) - 1(0 - 6) + (-2)(3 - 0) = 6 - 6 = 0$.

Answer: Yes; $|A| = 0$ (odd-order skew-symmetric).

Q6. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda = -1, 5$.

$\lambda = -1: [2 \ 4; 2 \ 4]X = 0 \Rightarrow x = -2y \Rightarrow (2, -1)$. $\lambda = 5: [-4 \ 4; 2 \ -2]X = 0 \Rightarrow x = y \Rightarrow (1, 1)$.

Answer: $\lambda = -1: (2, -1)$; $\lambda = 5: (1, 1)$.

Q7. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1, 3$.

$\lambda = 1: x + y = 0 \Rightarrow (1, -1)$. $\lambda = 3: -x + y = 0 \Rightarrow (1, 1)$.

Answer: $\lambda = 1: (1, -1)$; $\lambda = 3: (1, 1)$ (orthogonal, since A is symmetric).

Q8. Find the eigenvalues of $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$.

Lower triangular, so the eigenvalues are the diagonal entries.

Answer: 3, 4, 1.**Q9. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.**

$$\text{tr} = 7, S_2 = (5 - 1) + (1 - 9) + (5 - 1) = 0, |A| = 1(4) - 1(1 - 3) + 3(1 - 15) = 4 + 2 - 42 = -36.$$

$$\lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0 \Rightarrow (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$$

$$\lambda = -2: \text{rows } (3, 1, 3), (1, 7, 1) \Rightarrow x : y : z = (1 - 21) : (3 - 3) : (21 - 1) = (1, 0, -1).$$

$$\lambda = 3: \text{rows } (-2, 1, 3), (1, 2, 1) \Rightarrow (1 - 6) : (3 + 2) : (-4 - 1) = (1, -1, 1).$$

$$\lambda = 6: \text{rows } (-5, 1, 3), (1, -1, 1) \Rightarrow (1 + 3) : (3 + 5) : (5 - 1) = (1, 2, 1).$$

Answer: $\lambda = -2$: (1, 0, -1); $\lambda = 3$: (1, -1, 1); $\lambda = 6$: (1, 2, 1).**Q10. Two eigenvalues of a 3×3 matrix are 2 and 3 and its trace is 10. Find the third.**

$$\text{Sum of eigenvalues} = \text{trace} = 2 + 3 + \lambda_3 = 10.$$

Answer: $\lambda_3 = 5$.**Q11. A has eigenvalues 1, 2, 3. Find $|A|$ and the eigenvalues of A^{-1} and $A^2 + I$.**

$$|A| = 6. A^{-1}: 1, 1/2, 1/3. A^2 + I: 2, 5, 10.$$

Answer: $|A| = 6$; A^{-1} : 1, 1/2, 1/3; $A^2 + I$: 2, 5, 10.**Q12. Using Cayley-Hamilton, find A^{-1} for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.**

$$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow A^2 - 4A - 5I = O \Rightarrow A^{-1} = (A - 4I)/5.$$

Answer: $A^{-1} = (1/5)[-3 \ 4; 2 \ -1]$.**Q13. Write the characteristic equation of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.**

$$\text{tr} = 11. S_2 = (24 - 25) + (6 - 9) + (4 - 4) = -4. |A| = 1(24 - 25) - 2(12 - 15) + 3(10 - 12) = -1 + 6 - 6 = -1.$$

Answer: $\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$ (and $A^3 - 11A^2 - 4A + I = O$).**Q14. Show that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalisable.**

$$\lambda = 1 \text{ (AM} = 2\text{)}. A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has rank } 1, \text{ so GM} = 2 - 1 = 1 < \text{AM}.$$

Answer: Only one independent eigenvector (1, 0), so A is not diagonalisable.

5.14 MCQs: Module V

1. A square matrix A has an inverse iff

- (a) $|A| \neq 0$ (b) A is diagonal (c) A is symmetric (d) $|A| = 0$

□ (a) — Non-singular.

2. $(AB)^{-1}$ equals

- (a) $B^{-1}A^{-1}$ (b) $A^{-1}B^{-1}$ (c) BA (d) AB

□ (a) — Reversal law.



- 3.** In the Gauss-Jordan method, we reduce $[A \mid I]$ to
(a) $[I \mid A]$ (b) $[A^{-1} \mid I]$ (c) $[I \mid A^{-1}]$ (d) $[O \mid I]$
□ (c) — Same row operations applied to I.
- 4.** If a zero row appears in the left block during Gauss-Jordan, then
(a) A is orthogonal (b) A is singular (c) $A^{-1} = O$ (d) $A = I$
□ (b) — No inverse exists.
- 5.** The inverse of $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ is
(a) $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$
□ (b) — Invert diagonal entries.
- 6.** The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is
(a) $(1/(ad + bc))\begin{bmatrix} d & b \\ c & a \end{bmatrix}$ (b) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (c) $(1/(ad - bc))\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (d) $(ad - bc)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
□ (c) — Standard formula.
- 7.** $|A^{-1}|$ equals
(a) $1/|A|$ (b) $|A|^2$ (c) $-|A|$ (d) $|A|$
□ (a) — $\det(A) \cdot \det(A^{-1}) = \det(I) = 1$.
- 8.** A matrix is symmetric if
(a) $A^T = -A$ (b) $A^T = A$ (c) $A^T A = I$ (d) $A^2 = A$
□ (b) — Definition.
- 9.** A matrix is skew-symmetric if
(a) $A^T = A$ (b) $A^2 = I$ (c) $A^T = A^{-1}$ (d) $A^T = -A$
□ (d) — Definition.
- 10.** The diagonal elements of a skew-symmetric matrix are
(a) 0 (b) arbitrary (c) equal (d) 1
□ (a) — $a_{ii} = -a_{ii}$.
- 11.** The determinant of a skew-symmetric matrix of odd order is
(a) 1 (b) a perfect square (c) -1 (d) 0
□ (d) — $|A| = -|A|$.
- 12.** A matrix A is orthogonal if
(a) $|A| = 0$ (b) $A^T = A$ (c) $A^2 = I$ (d) $AA^T = I$
□ (d) — $A^{-1} = A^T$.
- 13.** The determinant of an orthogonal matrix is
(a) 0 (b) any real number (c) ± 1 (d) 2
□ (c) — $|A|^2 = 1$.
- 14.** The symmetric part of A is
(a) $(A + A^T)/2$ (b) AA^T (c) $A + A^T$ (d) $(A - A^T)/2$
□ (a) — Unique decomposition.
- 15.** For any matrix A, the product AA^T is always
(a) skew-symmetric (b) singular (c) orthogonal (d) symmetric
□ (d) — $(AA^T)^T = AA^T$.
- 16.** Which matrix is orthogonal?
(a) $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
□ (a) — Rotation matrix.
- 17.** A non-zero vector X is an eigenvector of A if
(a) $AX = 0$ (b) $AX = X^T$ (c) $AX = \lambda X$ (d) $A = \lambda X$
□ (c) — Definition.



- 18.** The characteristic equation of A is
(a) $|A + \lambda I| = 1$ (b) $|A| = 0$ (c) $A - \lambda = 0$ (d) $|A - \lambda I| = 0$
□ (d) — Non-trivial solution of $(A - \lambda I)X = 0$.
- 19.** The sum of the eigenvalues of A equals
(a) order of A (b) $|A|$ (c) trace of A (d) rank of A
□ (c) — Property 1.
- 20.** The product of the eigenvalues of A equals
(a) $|A|$ (b) trace of A (c) rank of A (d) 1
□ (a) — Property 2.
- 21.** If 0 is an eigenvalue of A, then A is
(a) singular (b) symmetric (c) non-singular (d) orthogonal
□ (a) — Product of eigenvalues = $|A| = 0$.
- 22.** The eigenvalues of $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ are
(a) 0, 8 (b) 3, 5 (c) 15, 1 (d) 8, 2
□ (b) — Diagonal matrix.
- 23.** The eigenvalues of the triangular matrix $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 3 & 1 \\ 0 & 0 & -4 \end{bmatrix}$ are
(a) 0, 0, 0 (b) 7, 1, -4 (c) 2, 5, 7 (d) 2, 3, -4
□ (d) — Diagonal entries.
- 24.** The eigenvalues of $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ are
(a) 1, 6 (b) 2, 5 (c) 3, 4 (d) -2, -5
□ (b) — $\lambda^2 - 7\lambda + 10 = 0$.
- 25.** The eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are
(a) 0, 2 (b) 1, 1 (c) 2, 2 (d) 0, 1
□ (a) — trace 2, det 0.
- 26.** The characteristic equation of a 2×2 matrix A is
(a) $\lambda^2 + \text{tr}(A)\lambda + |A| = 0$ (b) $\lambda^2 - \text{tr}(A)\lambda + |A| = 0$ (c) $\lambda^2 - |A|\lambda + \text{tr } A = 0$ (d) $\lambda^2 - 1 = 0$
□ (b) — Shortcut.
- 27.** If λ is an eigenvalue of A, then the eigenvalue of A^{-1} is
(a) $1/\lambda$ (b) $-\lambda$ (c) λ (d) λ^2
□ (a) — Property 5.
- 28.** If λ is an eigenvalue of A, then the eigenvalue of A^3 is
(a) 3λ (b) $\lambda + 3$ (c) $1/\lambda^3$ (d) λ^3
□ (d) — $A^3X = \lambda^3X$.
- 29.** If the eigenvalues of A are 1, 2, 3, then the eigenvalues of $A + 2I$ are
(a) 1, 2, 3 (b) 2, 4, 6 (c) 3, 4, 5 (d) -1, 0, 1
□ (c) — $\lambda + 2$.
- 30.** If the eigenvalues of A are 2, 3, 4, then $|A|$ equals
(a) 6 (b) 24 (c) 9 (d) 12
□ (b) — Product.
- 31.** If the eigenvalues of A are 2, 3, 4, then the eigenvalues of $\text{adj } A$ are
(a) 24, 24, 24 (b) 2, 3, 4 (c) $1/2, 1/3, 1/4$ (d) 12, 8, 6
□ (d) — $|A|/\lambda = 24/\lambda$.
- 32.** A and A^T have
(a) reciprocal eigenvalues (b) no relation (c) the same eigenvalues (d) negative eigenvalues
□ (c) — Same characteristic polynomial.



- 33.** The eigenvalues of a real symmetric matrix are always
(a) complex (b) purely imaginary (c) zero (d) real
□ (d) — Property 6.
- 34.** The eigenvalues of a real skew-symmetric matrix are
(a) always 1 (b) real and non-zero (c) zero or purely imaginary (d) always positive
□ (c) — Property 6.
- 35.** The eigenvalues of an orthogonal matrix have modulus
(a) 1 (b) 0 (c) any value (d) 2
□ (a) — Length is preserved.
- 36.** Eigenvectors of a real symmetric matrix corresponding to distinct eigenvalues are
(a) orthogonal (b) equal (c) zero (d) parallel
□ (a) — Property 7.
- 37.** Eigenvectors corresponding to distinct eigenvalues are always
(a) equal (b) LI (c) orthogonal (d) LD
□ (b) — Always LI (orthogonal only if symmetric).
- 38.** If X is an eigenvector of A , then so is
(a) $X + 1$ (b) X^2 (c) X^T (d) kX for any non-zero k
□ (d) — $A(kX) = \lambda(kX)$.
- 39.** The geometric multiplicity of an eigenvalue is
(a) always equal to n (b) at most its algebraic multiplicity (c) greater than its algebraic multiplicity (d) always 1
□ (b) — $1 \leq GM \leq AM$.
- 40.** Cayley-Hamilton theorem states that every square matrix
(a) is diagonalisable (b) satisfies its own characteristic equation (c) has real eigenvalues (d) is invertible
□ (b) — Statement.
- 41.** For $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, Cayley-Hamilton gives
(a) $A^2 - 3I = O$ (b) $A^2 + 2A - 3I = O$ (c) $A^2 - 2A - 3I = O$ (d) $A^2 - 2A + 3I = O$
□ (c) — $\text{tr} = 2, |A| = -3$.
- 42.** If $A^2 - 5A + 6I = O$, then A^{-1} equals
(a) $(A + 5I)/6$ (b) $6(A - 5I)$ (c) $(5I - A)/6$ (d) $(A - 5I)/6$
□ (c) — $A - 5I + 6A^{-1} = O$.
- 43.** Cayley-Hamilton theorem can be used to find
(a) A^{-1} and higher powers of A (b) eigenvectors directly (c) the rank only (d) the trace only
□ (a) — Standard uses.
- 44.** If $A^2 = I$, the possible eigenvalues of A are
(a) 0 only (b) $\pm i$ (c) 2 (d) ± 1
□ (d) — $\lambda^2 = 1$.
- 45.** A matrix with $A^2 = A$ (idempotent) has eigenvalues
(a) any value (b) ± 1 (c) only 2 (d) 0 or 1
□ (d) — $\lambda^2 = \lambda$.
- 46.** If A is similar to B ($B = P^{-1}AP$), then
(a) they have the same eigenvalues (b) they have the same eigenvectors (c) $|A| \neq |B|$ (d) $A = B$
□ (a) — Same characteristic polynomial.



- 47.** A is diagonalisable by P if the columns of P are
(a) rows of A (b) zero vectors (c) eigenvectors of A (d) eigenvalues of A
□ (c) — $P^{-1}AP = D$.
- 48.** For a 3×3 matrix, S_2 in the characteristic equation is
(a) sum of principal minors of order 2 (b) determinant (c) trace squared (d) sum of diagonal elements
□ (a) — $\lambda^3 - (\text{tr})\lambda^2 + S_2\lambda - |A| = 0$.
- 49.** The eigenvalues of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are
(a) 0, 6, 12 (b) 1, 3, 5 (c) 2, 3, 13 (d) 0, 3, 15
□ (d) — $\text{tr} = 18, |A| = 0, S_2 = 45$.
- 50.** The eigenvalues of the rotation matrix are
(a) ± 1 (b) 0, 1 (c) $\cos \theta, \sin \theta$ (d) $e^{\pm i\theta}$
□ (d) — $\lambda^2 - 2\lambda \cos \theta + 1 = 0$.

Additional Solved Problems (All Modules)

These extra problems cover question types that often appear in the university exam but were not fully covered above. Work each one yourself first, then compare.

Module I

Problem 1. Find the surface area generated by revolving the parabola $y = \sqrt{x}$, $0 \leq x \leq 2$, about the x-axis.

$$y' = 1/(2\sqrt{x}). \quad 1 + y'^2 = 1 + 1/(4x) = (4x + 1)/(4x). \quad y\sqrt{1 + y'^2} = \sqrt{x} \cdot \sqrt{(4x + 1)/(4x)} = (1/2)\sqrt{4x + 1}.$$

$$S = 2\pi \int_0^2 (1/2)\sqrt{4x + 1} \, dx = \pi [(4x + 1)^{3/2}/6]_0^2 = (\pi/6)(27 - 1).$$

Answer: $S = 13\pi/3$ square units.

Problem 2. Find the volume generated when the region bounded by $y = 4 - x^2$ and the x-axis is rotated about the x-axis.

$$\text{The parabola meets the x-axis at } x = \pm 2. \quad V = \pi \int_{-2}^2 (4 - x^2)^2 \, dx = 2\pi \int_0^2 (16 - 8x^2 + x^4) \, dx.$$

$$= 2\pi [16x - 8x^3/3 + x^5/5]_0^2 = 2\pi (32 - 64/3 + 32/5) = 2\pi (480 - 320 + 96)/15 = 2\pi(256/15).$$

Answer: $V = 512\pi/15$ cubic units.

Problem 3. Evaluate $\int_0^1 x^2 \sqrt{1 - x} \, dx$ using Beta function.

$$= B(3, 3/2) = \Gamma(3)\Gamma(3/2)/\Gamma(9/2). \quad \Gamma(9/2) = (7/2)(5/2)(3/2)\Gamma(3/2).$$

$$= 2/((7/2)(5/2)(3/2)) = 2/(105/8) = 16/105.$$

Answer: $16/105$.

Problem 4. Evaluate $\int_0^\infty x^3 e^{-2x} \, dx$.

$$= \Gamma(4)/2^4 = 6/16.$$

Answer: $3/8$.

Problem 5. Test the convergence of $\int_e^\infty dx/(x (\log x)^2)$.

Put $t = \log x$, $dt = dx/x$. When $x = e$, $t = 1$; when $x \rightarrow \infty$, $t \rightarrow \infty$.

$$\text{Integral} = \int_1^\infty dt/t^2 = 1 \quad (p = 2 > 1).$$

Answer: Converges to 1.

Problem 6. Prove that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} \, dx$ and hence evaluate $\int_0^\infty x^4 e^{-x^2} \, dx$.

$$\text{In } \Gamma(n) = \int_0^\infty e^{-t} t^{n-1} \, dt, \text{ put } t = x^2, \, dt = 2x \, dx: \Gamma(n) = \int_0^\infty e^{-x^2} x^{2n-2} \cdot 2x \, dx = 2\int_0^\infty e^{-x^2} x^{2n-1} \, dx.$$

$$\text{For } x^4: 2n - 1 = 4 \Rightarrow n = 5/2. \quad \text{Integral} = \Gamma(5/2)/2 = (3\sqrt{\pi}/4)/2.$$

Answer: $3\sqrt{\pi}/8$.

Problem 7. Evaluate $\int_0^{2a} x^2 \sqrt{2ax - x^2} \, dx$.



Put $x = 2a \sin^2\theta$, $dx = 4a \sin \theta \cos \theta d\theta$. $2ax - x^2 = 4a^2 \sin^2\theta(1 - \sin^2\theta) = 4a^2 \sin^2\theta \cos^2\theta$.

Integral $= \int_0^{\pi/2} 4a^2 \sin^4\theta \cdot 2a \sin \theta \cos \theta \cdot 4a \sin \theta \cos \theta d\theta = 32a^4 \int_0^{\pi/2} \sin^6\theta \cos^2\theta d\theta$.

By Wallis/Beta: $\int \sin^6\theta \cos^2\theta d\theta = (5 \cdot 3 \cdot 1)(1)/(8 \cdot 6 \cdot 4 \cdot 2) \cdot \pi/2 = 15\pi/768 = 5\pi/256$.

Answer: $32a^4 \times 5\pi/256 = 5\pi a^4/8$.

Module II

Problem 1. Expand $e^x \sin x$ by Maclaurin's series up to x^5 .

Multiply the series: $(1 + x + x^2/2 + x^3/6 + x^4/24)(x - x^3/6 + x^5/120)$.

x-terms: x . x^2 : x^2 . x^3 : $x^3/2 - x^3/6 = x^3/3$. x^4 : $x^4/6 - x^4/6 = 0$. x^5 : $x^5/24 - x^5/12 + x^5/120 = (5 - 10 + 1)x^5/120 = -x^5/30$.

Answer: $e^x \sin x = x + x^2 + x^3/3 - x^5/30 + \dots$

Problem 2. Expand $\log \sec x$ by Maclaurin's series up to x^6 .

Let $y = \log \sec x$; $y' = \tan x = x + x^3/3 + 2x^5/15 + \dots$

Integrate term by term ($y(0) = 0$): $y = x^2/2 + x^4/12 + x^6/45 + \dots$

Answer: $\log \sec x = x^2/2 + x^4/12 + x^6/45 + \dots$

Problem 3. Expand $\sqrt{1+x}$ up to x^3 and find $\sqrt{1.1}$.

$(1+x)^{1/2} = 1 + x/2 - x^2/8 + x^3/16 - \dots$

For $x = 0.1$: $1 + 0.05 - 0.00125 + 0.0000625 = 1.0488125$.

Answer: $\sqrt{1.1} \approx 1.0488$ (true value 1.04881).

Problem 4. Show that $\sin x < x$ for $x > 0$ using LMVT.

Let $f(t) = t - \sin t$ on $[0, x]$. $f'(t) = 1 - \cos t \geq 0$ with equality only at isolated points, so f is strictly increasing.

Hence $f(x) > f(0) = 0$.

Answer: $x - \sin x > 0$, i.e. $\sin x < x$ for $x > 0$.

Problem 5. A window is a rectangle surmounted by a semicircle. If the perimeter is 10 m, find the dimensions that admit maximum light.

Let width $2r$ and rectangle height h . Perimeter: $2r + 2h + \pi r = 10 \Rightarrow h = (10 - 2r - \pi r)/2$.

Area $A = 2rh + \pi r^2/2 = r(10 - 2r - \pi r) + \pi r^2/2 = 10r - 2r^2 - \pi r^2/2$.

$A'(r) = 10 - 4r - \pi r = 0 \Rightarrow r = 10/(4 + \pi)$. $A'' = -(4 + \pi) < 0 \Rightarrow$ maximum.

$h = (10 - r(2 + \pi))/2 = [10(4 + \pi) - 10(2 + \pi)]/[2(4 + \pi)] = 10/(4 + \pi) = r$.

Answer: $r = h = 10/(4 + \pi) \approx 1.40$ m; width $2r \approx 2.80$ m.

Module III

Problem 1. Find the stationary points of $f(x, y) = x^3 - 3x + y^2 + 2y$ and classify them.

$$f_x = 3x^2 - 3 = 0 \Rightarrow x = \pm 1. f_y = 2y + 2 = 0 \Rightarrow y = -1.$$

$$r = 6x, s = 0, t = 2. \text{ At } (1, -1): D = 12 > 0, r > 0 \Rightarrow \text{minimum, } f = 1 - 3 + 1 - 2 = -3.$$

$$\text{At } (-1, -1): D = -12 < 0 \Rightarrow \text{saddle point.}$$

Answer: Minimum -3 at (1, -1); saddle point at (-1, -1).

Problem 2. Find the minimum value of $f(x, y) = xy + 50/x + 20/y$ for $x, y > 0$.

$$f_x = y - 50/x^2 = 0 \Rightarrow y = 50/x^2. f_y = x - 20/y^2 = 0 \Rightarrow x = 20/y^2.$$

$$\text{Substitute: } x = 20x^4/2500 \Rightarrow x^3 = 125 \Rightarrow x = 5, y = 2.$$

$$r = 100/x^3 = 0.8, t = 40/y^3 = 5, s = 1. D = 4 - 1 = 3 > 0, r > 0 \Rightarrow \text{minimum.}$$

Answer: Minimum value $f(5, 2) = 10 + 10 + 10 = 30$.

Problem 3. Find the volume of the largest rectangular box that can be inscribed in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Box with corner (x, y, z) in the first octant has $V = 8xyz$. $F = 8xyz + \lambda(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1)$.

$$8yz + 2\lambda x/a^2 = 0, 8xz + 2\lambda y/b^2 = 0, 8xy + 2\lambda z/c^2 = 0. \text{ Multiply by } x, y, z: 8xyz = -2\lambda x^2/a^2 = -2\lambda y^2/b^2 = -2\lambda z^2/c^2.$$

$$\text{So } x^2/a^2 = y^2/b^2 = z^2/c^2 = 1/3 \Rightarrow x = a/\sqrt{3}, y = b/\sqrt{3}, z = c/\sqrt{3}.$$

Answer: $V_{\max} = 8abc/(3\sqrt{3})$.

Problem 4. If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\partial(u, v, w)/\partial(x, y, z)$.

$$\text{Rows: } u: (2x, -2, 0); v: (1, 1, 1); w: (1, -2, 3).$$

$$J = 2x(3 + 2) - (-2)(3 - 1) + 0 = 10x + 4.$$

Answer: $J = 10x + 4$.

Problem 5. If $z = f(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$, show that $z_u^2 + z_v^2 = e^{2u}(z_x^2 + z_y^2)$.

$$z_u = z_x x_u + z_y y_u = x z_x + y z_y. z_v = z_x(-y) + z_y x = -y z_x + x z_y.$$

$$z_u^2 + z_v^2 = x^2 z_x^2 + 2xy z_x z_y + y^2 z_y^2 + y^2 z_x^2 - 2xy z_x z_y + x^2 z_y^2 = (x^2 + y^2)(z_x^2 + z_y^2).$$

Answer: Since $x^2 + y^2 = e^{2u}$, the result follows.

Module IV and V

Problem 1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ -1 & 3 \end{bmatrix}$.

$$\text{tr } A = 11. S_2 = (15 - 1) + (9 - 1) + (15 - 1) = 36. |A| = 3(15 - 1) + 1(-3 + 1) + 1(1 - 5) = 42 - 2 - 4 = 36.$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0 \Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 6) = 0.$$

$$\lambda = 2: \text{ rows } (1, -1, 1), (-1, 3, -1) \Rightarrow x : y : z = (1 - 3) : (-1 + 1) : (3 - 1) = -2 : 0 : 2 \Rightarrow (1, 0, -1).$$



$\lambda = 3$: rows $(0, -1, 1), (-1, 2, -1) \Rightarrow x : y : z = (1 - 2) : (-1 - 0) : (0 - 1) \Rightarrow (1, 1, 1)$.

$\lambda = 6$: rows $(-3, -1, 1), (-1, -1, -1) \Rightarrow x : y : z = (1 + 1) : (-1 - 3) : (3 - 1) = 2 : -4 : 2 \Rightarrow (1, -2, 1)$.

Answer: $\lambda = 2, 3, 6$ with orthogonal eigenvectors $(1, 0, -1), (1, 1, 1), (1, -2, 1)$.

Problem 2. Using Cayley-Hamilton, find A^{-1} for $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$\text{tr } A = -1$. $S_2 = (1 - 0) + (-1 - 0) + (-1 - 4) = -5$. $|A| = -1 \times (-1 - 4) = 5$.

Characteristic equation: $\lambda^3 + \lambda^2 - 5\lambda - 5 = 0 \Rightarrow A^3 + A^2 - 5A - 5I = O \Rightarrow A^{-1} = (A^2 + A - 5I)/5$.

$A^2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $A^2 + A - 5I = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$.

Answer: $A^{-1} = (1/5)\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$.

Problem 3. Show that the vectors $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ span $\mathbb{R}^3$ and write $(2, 3, 4)$ in terms of them.

$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(0 - 1) - 1(1 - 0) + 0 = -2 \neq 0 \Rightarrow \text{LI} \Rightarrow \text{basis, hence they span } \mathbb{R}^3$.

$a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1) = (2, 3, 4)$: $a + b = 2, a + c = 3, b + c = 4$. Adding: $a + b + c = 4.5 \Rightarrow c = 2.5, b = 1.5, a = 0.5$.

Answer: $(2, 3, 4) = 0.5(1, 1, 0) + 1.5(1, 0, 1) + 2.5(0, 1, 1)$.

Problem 4. Find the value of k for which the vectors $(1, 2, 1), (k, 1, 1), (0, 3, 1)$ are linearly dependent.

$\begin{vmatrix} 1 & 2 & 1 \\ k & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 1(1 - 3) - 2(k - 0) + 1(3k - 0) = -2 - 2k + 3k = k - 2$.

Answer: LD when $k = 2$.

Short-Answer Questions (2 marks each) with Answers

These are the kind of questions asked in Part-A (short answer) of the university examination. Answer each in 3-5 lines.

Module I

Q1. Define an improper integral of the first kind and give an example.

An integral $\int_a^b f(x) dx$ in which one or both limits are infinite, e.g. $\int_1^\infty dx/x^2$. It is defined as $\lim_{t \rightarrow \infty} \int_1^t dx/x^2 = 1$.

Q2. Define an improper integral of the second kind and give an example.

An integral over a finite interval in which the integrand is unbounded at some point, e.g. $\int_0^1 dx/\sqrt{x}$ (unbounded at $x = 0$). Its value is $\lim_{\varepsilon \rightarrow 0} [2\sqrt{x}]_\varepsilon^1 = 2$.

Q3. State the p-test for $\int_1^\infty dx/x^p$.

It converges if $p > 1$ (to $1/(p - 1)$) and diverges if $p \leq 1$.

Q4. Define the Gamma function and state its recurrence relation.

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ for $n > 0$. Recurrence: $\Gamma(n + 1) = n\Gamma(n)$; for positive integers, $\Gamma(n + 1) = n!$.

Q5. Define the Beta function and give its relation to the Gamma function.

$B(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1} dx$, $m, n > 0$. $B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m + n)$.

Q6. Prove that $B(m, n) = B(n, m)$.

Put $x = 1 - t$ in the definition: $B(m, n) = \int_1^0 (1 - t)^{m-1} t^{n-1} (-dt) = \int_0^1 t^{n-1}(1 - t)^{m-1} dt = B(n, m)$.

Q7. Find the value of $\Gamma(1/2)$.

$\Gamma(1/2) = \sqrt{\pi}$ (put $x = t^2$ to get $2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}$).

Q8. Evaluate $\int_0^\infty x^5 e^{-x} dx$.

$= \Gamma(6) = 5! = 120$.

Q9. Write the formula for the length of a polar curve $r = f(\theta)$.

$s = \int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2} d\theta$.

Q10. Write the formula for the volume of a solid of revolution about the x-axis.

$V = \int_a^b \pi y^2 dx$ (disc method).

Q11. Write the formula for the surface area generated when $y = f(x)$ revolves about the x-axis.

$S = \int_a^b 2\pi y \sqrt{1 + (dy/dx)^2} dx$.



Q12. Evaluate $\int_0^{\pi/2} \sin^5 x \cos^3 x \, dx$ using Beta function.

$$= (1/2)B(3, 2) = (1/2)\Gamma(3)\Gamma(2)/\Gamma(5) = (1/2)(2 \times 1)/24 = 1/24.$$

Module II

Q1. State Rolle's theorem.

If f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Q2. State Lagrange's Mean Value Theorem and give its geometrical meaning.

If f is continuous on $[a, b]$ and differentiable on (a, b) , there exists $c \in (a, b)$ with $f'(c) = [f(b) - f(a)]/(b - a)$. Geometrically, the tangent at c is parallel to the chord joining the end points.

Q3. Why is Rolle's theorem not applicable to $f(x) = |x|$ in $[-1, 1]$?

f is not differentiable at $x = 0 \in (-1, 1)$ (left derivative $-1 \neq$ right derivative 1).

Q4. State Cauchy's Mean Value Theorem.

If f, g are continuous on $[a, b]$, differentiable on (a, b) and $g'(x) \neq 0$ there, then $[f(b) - f(a)]/[g(b) - g(a)] = f'(c)/g'(c)$ for some $c \in (a, b)$.

Q5. Find c of LMVT for $f(x) = x^2 - 3x$ on $[1, 4]$.

$$[f(4) - f(1)]/3 = [4 - (-2)]/3 = 2. \quad f'(c) = 2c - 3 = 2 \Rightarrow c = 5/2.$$

Q6. State the first derivative test for local extrema.

At a critical point c , if f' changes from $+$ to $-$, f has a local maximum; from $-$ to $+$, a local minimum; no change of sign, no extremum.

Q7. Define the radius of convergence of a power series.

The number R such that $\sum c_n(x - a)^n$ converges absolutely for $|x - a| < R$ and diverges for $|x - a| > R$. $R = \lim |c_n/c_{n+1}|$.

Q8. Write Maclaurin's series for $f(x)$.

$$f(x) = f(0) + x f'(0) + (x^2/2!) f''(0) + (x^3/3!) f'''(0) + \dots$$

Q9. Write the Maclaurin series of $\sin x$ and $\cos x$.

$$\sin x = x - x^3/3! + x^5/5! - \dots; \quad \cos x = 1 - x^2/2! + x^4/4! - \dots \text{ (valid for all } x\text{)}.$$

Q10. Write Taylor's theorem with Lagrange's form of remainder.

$$f(a + h) = f(a) + hf'(a) + \dots + (h^n/n!)f^{(n)}(a) + (h^{n+1}/(n+1)!)f^{(n+1)}(a + \theta h), \quad 0 < \theta < 1.$$

Q11. Find the radius of convergence of $\sum x^n/n^2$.

$$R = \lim (n+1)^2/n^2 = 1.$$



Module III

Q1. If $u = x^2y + y^2x$, find u_x and u_y .

$$u_x = 2xy + y^2, u_y = x^2 + 2xy.$$

Q2. State Euler's theorem on homogeneous functions.

If $f(x, y)$ is homogeneous of degree n , then $x \partial f / \partial x + y \partial f / \partial y = nf$.

Q3. Define the Jacobian of u, v with respect to x, y .

$$\partial(u, v) / \partial(x, y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

Q4. If $x = r \cos \theta$, $y = r \sin \theta$, find $\partial(x, y) / \partial(r, \theta)$.

$$= \cos \theta \cdot r \cos \theta - (-r \sin \theta) \sin \theta = r.$$

Q5. When are two functions u and v said to be functionally dependent?

When there is a relation $F(u, v) = 0$; this happens iff $\partial(u, v) / \partial(x, y) = 0$ identically.

Q6. Define the Hessian matrix of $f(x, y)$.

$H = [f_{xx} \ f_{xy}; f_{yx} \ f_{yy}]$, the matrix of second partial derivatives. It is symmetric.

Q7. Define a saddle point.

A stationary point where f has neither a maximum nor a minimum; f increases in some directions and decreases in others. Condition: $rt - s^2 < 0$.

Q8. State the conditions for $f(x, y)$ to have a maximum at (a, b) .

$f_x = f_y = 0$ at (a, b) , $rt - s^2 > 0$ and $r < 0$, where $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$.

Q9. Find the stationary point of $f(x, y) = x^2 + y^2 - 2x + 4y$.

$f_x = 2x - 2 = 0$, $f_y = 2y + 4 = 0 \Rightarrow (1, -2)$. It is a minimum ($r = 2 > 0$, $D = 4 > 0$) with value -5 .

Q10. What is the Lagrange multiplier method used for?

To find extreme values of a function $f(x, y, z)$ subject to a constraint $\varphi(x, y, z) = 0$, by forming $F = f + \lambda \varphi$ and solving $F_x = F_y = F_z = 0$, $\varphi = 0$.

Q11. Write the Jacobian for spherical polar coordinates.

$$\partial(x, y, z) / \partial(r, \theta, \varphi) = r^2 \sin \theta.$$

Module IV

Q1. Define a vector space.

A non-empty set V with vector addition and scalar multiplication that is closed under both and satisfies the axioms of commutativity, associativity, zero vector, additive inverse, distributive laws and $1 \cdot v = v$.



Q2. Define a subspace and give an example.

A non-empty subset W of V that is itself a vector space under the same operations (closed under addition and scalar multiplication). Example: $\{(x, y, 0)\}$ in $\mathbb{R}^3$.

Q3. Define linear independence of vectors.

$v_1, \dots, v_k$ are LI if $c_1 v_1 + \dots + c_k v_k = 0$ implies all $c_i = 0$.

Q4. Define basis and dimension.

A basis is an LI set that spans V . The number of vectors in any basis is the dimension of V .

Q5. What is the dimension of the vector space of 2×2 matrices?

4 (basis: the four matrices with a single entry 1 and the rest 0).

Q6. Define the rank of a matrix.

The order of the largest non-zero minor; equivalently, the number of non-zero rows in its row echelon form.

Q7. State the condition for consistency of $AX = B$.

The system is consistent iff $\text{rank } A = \text{rank } [A \mid B]$.

Q8. When does a homogeneous system $AX = 0$ of n equations in n unknowns have non-trivial solutions?

When $|A| = 0$ (equivalently $\text{rank } A < n$).

Q9. State the rank-nullity theorem.

For an $m \times n$ matrix A , $\text{rank}(A) + \text{nullity}(A) = n$.

Q10. Find the rank of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

The rows are proportional and $|A| = 0$, but $A \neq O$, so $\text{rank} = 1$.

Q11. Are (1, 2, 3) and (2, 4, 6) linearly independent?

No; $(2, 4, 6) = 2(1, 2, 3)$, so they are LD.

Module V

Q1. State the Cayley-Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

Q2. Define eigenvalue and eigenvector.

λ is an eigenvalue of A if $AX = \lambda X$ for some non-zero vector X ; X is the corresponding eigenvector.



Q3. Write any two properties of eigenvalues.

(i) Sum of eigenvalues = trace of A. (ii) Product of eigenvalues = $|A|$.

Q4. Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

A is triangular; eigenvalues 1 and 3.

Q5. If 3 is an eigenvalue of A, find an eigenvalue of A^{-1} and A^2 .

$1/3$ and 9.

Q6. Define an orthogonal matrix and write its determinant.

A square matrix with $AA^T = A^T A = I$ (so $A^{-1} = A^T$). Its determinant is ± 1 .

Q7. Show that the diagonal elements of a skew-symmetric matrix are zero.

$a_{ij} = -a_{ji}$ for all i, j . For $i = j$: $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$.

Q8. Write the characteristic equation of $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

$\lambda^2 - 6\lambda + 5 = 0$ (trace 6, determinant 5); eigenvalues 1 and 5.

Q9. If $A^2 - 3A + 2I = O$, find A^{-1} .

Multiply by A^{-1} : $A - 3I + 2A^{-1} = O \Rightarrow A^{-1} = (3I - A)/2$.

Q10. What is the Gauss-Jordan method for finding the inverse?

Reduce $[A \mid I]$ by elementary row operations to $[I \mid B]$; then $B = A^{-1}$.

Q11. Express $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$ as the sum of symmetric and skew-symmetric matrices.

$P = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; $P + Q = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$.



Model Question Paper with Solutions

Practice pattern · Time: 3 hours · Part A: 10 short questions (2 marks each) · Part B: long questions. Follow the exact pattern announced by your university.

Note: This is a practice paper written to match the syllabus. It is not an official university question paper. Use it to time yourself: 3 hours, then check against the solutions.

Part A (2 marks each)

Q1. Evaluate $\Gamma(7/2)$.

$$\Gamma(7/2) = (5/2)(3/2)(1/2)\sqrt{\pi} = 15\sqrt{\pi}/8.$$

Q2. Test the convergence of $\int_1^\infty dx/x^3$.

$p = 3 > 1$, so it converges (value $1/2$).

Q3. Verify Rolle's theorem for $f(x) = x^2 - 4$ on $[-2, 2]$.

Polynomial, $f(-2) = f(2) = 0$, $f'(c) = 2c = 0 \Rightarrow c = 0 \in (-2, 2)$.

Q4. Write the Maclaurin series of e^{-x} up to x^3 .

$$1 - x + x^2/2 - x^3/6.$$

Q5. If $u = x^3y^2$, find u_{xy} .

$$u_x = 3x^2y^2, u_{xy} = 6x^2y.$$

Q6. Find $\partial(u, v)/\partial(x, y)$ for $u = 2x - y$, $v = x + 3y$.

$$(2)(3) - (-1)(1) = 7.$$

Q7. Is $\{(1, 1), (2, 2)\}$ a basis of $\mathbb{R}^2$?

No; the vectors are LD ($(2, 2) = 2(1, 1)$), so they cannot form a basis.

Q8. Find the rank of $[1 \ 2 \ 3; 2 \ 4 \ 6]$.

$$R_2 - 2R_1 = 0, \text{ so rank} = 1.$$

Q9. Find the eigenvalues of $[3 \ 1; 1 \ 3]$.

$$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda = 2, 4.$$

Q10. State the Cayley-Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

Part B (long answer)

Problem 1. (a) Find the length of the curve $y = x^2/2$ from $x = 0$ to $x = 1$. (b) Evaluate $\int_0^\infty x^2 e^{-x^2} dx$.

$$(a) y' = x, s = \int_0^1 \sqrt{1+x^2} dx = \left[\frac{x}{2} \sqrt{1+x^2} + \frac{1}{2} \log(x + \sqrt{1+x^2}) \right]_0^1 = \frac{\sqrt{2}}{2} + \frac{1}{2} \log(1 + \sqrt{2}).$$

$$(b) \text{ Put } x^2 = t: \int_0^\infty t e^{-t} dt / (2\sqrt{t}) = (1/2) \int_0^\infty t^{1/2} e^{-t} dt = (1/2) \Gamma(3/2) = \sqrt{\pi}/4.$$

Answer: (a) $(1/2)[\sqrt{2} + \log(1 + \sqrt{2})] \approx 1.148$; (b) $\sqrt{\pi}/4$.

Problem 2. Show that $\int_0^{\pi/2} \sqrt{\sin \theta} d\theta \times \int_0^{\pi/2} d\theta / \sqrt{\sin \theta} = \pi$.

$$\text{First integral} = (1/2)B(3/4, 1/2) = \Gamma(3/4)\Gamma(1/2)/(2\Gamma(5/4)). \text{ Second} = (1/2)B(1/4, 1/2) = \Gamma(1/4)\Gamma(1/2)/(2\Gamma(3/4)).$$

$$\text{Product} = \Gamma(1/2)^2 \Gamma(1/4)/(4\Gamma(5/4)) = \pi \Gamma(1/4)/(4 \cdot (1/4)\Gamma(1/4)) = \pi.$$

Answer: Proved.

Problem 3. Find the volume of the solid generated by revolving the loop of the curve $y^2 = x(x-1)^2$ ($0 \leq x \leq 1$) about the x-axis.

$$V = \pi \int_0^1 y^2 dx = \pi \int_0^1 x(x-1)^2 dx = \pi \int_0^1 (x^3 - 2x^2 + x) dx = \pi(1/4 - 2/3 + 1/2).$$

Answer: $V = \pi/12$ cubic units.

Problem 4. Verify Lagrange's Mean Value Theorem for $f(x) = x(x-1)(x-2)$ on $[0, 1/2]$.

$$f \text{ is a polynomial. } f(0) = 0; f(1/2) = (1/2)(-1/2)(-3/2) = 3/8. \text{ Slope of chord} = (3/8)/(1/2) = 3/4.$$

$$f(x) = x^3 - 3x^2 + 2x \Rightarrow f'(x) = 3x^2 - 6x + 2 = 3/4 \Rightarrow 12x^2 - 24x + 5 = 0 \Rightarrow x = 1 \pm \sqrt{21}/6.$$

Answer: $c = 1 - \sqrt{21}/6 \approx 0.236 \in (0, 1/2)$. Verified.

Problem 5. Expand $\log(1 + e^x)$ by Maclaurin's series up to x^4 .

$$y = \log(1 + e^x); y(0) = \log 2. y' = e^x/(1 + e^x), y'(0) = 1/2.$$

$$y'' = e^x/(1 + e^x)^2 = y'(1 - y'), y''(0) = 1/4.$$

$$y''' = y''(1 - 2y'), y'''(0) = (1/4)(0) = 0. y'''' = y'''(1 - 2y') - 2(y'')^2, y''''(0) = 0 - 2/16 = -1/8.$$

$$\text{Series: } \log 2 + x/2 + x^2/8 + 0 - (1/8)x^4/24.$$

Answer: $\log(1 + e^x) = \log 2 + x/2 + x^2/8 - x^4/192 + \dots$

Problem 6. Find the extreme values of $f(x, y) = x^2 + y^2 + 6x + 12$.

$$f_x = 2x + 6 = 0, f_y = 2y = 0 \Rightarrow (-3, 0). r = 2, s = 0, t = 2, D = 4 > 0, r > 0.$$

Answer: Minimum value $f(-3, 0) = 9 - 18 + 12 = 3$.

Problem 7. Find the maximum value of $u = x^2 y^3 z^4$ subject to $2x + 3y + 4z = a$.

$$F = 2 \log x + 3 \log y + 4 \log z + \lambda(2x + 3y + 4z - a). 2/x + 2\lambda = 0, 3/y + 3\lambda = 0, 4/z + 4\lambda = 0 \Rightarrow x = y = z = -1/\lambda.$$

$$9x = a \Rightarrow x = y = z = a/9.$$

Answer: $u_{\max} = (a/9)^{2+3+4} = (a/9)^9$.

Problem 8. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2v$.

$$z = uvw; y = uv - uvw = uv(1 - w); x = u - uv = u(1 - v).$$

Matrix of partials: $x: (1 - v, -u, 0); y: (v(1 - w), u(1 - w), -uv); z: (vw, uw, uv)$.

Add rows: $R_1 + R_2 + R_3 = (1, 0, 0)$. Replacing R_1 by this sum: $J = \begin{vmatrix} 1 & 0 & 0 \\ v(1 - w) & u(1 - w) & -uv \\ vw & uw & uv \end{vmatrix} = u(1 - w)uv + uv \cdot uw = u^2v(1 - w + w)$.

Answer: $J = u^2v$.

Problem 9. Test the consistency and solve: $x + y + z = 6$, $x - y + 2z = 5$, $3x + y + z = 8$.

$R_2 - R_1: (0, -2, 1 \mid -1)$. $R_3 - 3R_1: (0, -2, -2 \mid -10)$. $R_3 - R_2: (0, 0, -3 \mid -9) \Rightarrow z = 3$.

$$-2y + 3 = -1 \Rightarrow y = 2. x = 6 - 2 - 3 = 1. \text{rank } A = \text{rank } [A|B] = 3.$$

Answer: Consistent; unique solution $x = 1, y = 2, z = 3$.

Problem 10. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ by the Gauss-Jordan method.

$R_2 - R_1: [0 \ 1 \ 2 \mid -1 \ 1 \ 0]; R_3 - R_1: [0 \ 3 \ 8 \mid -1 \ 0 \ 1]$. $R_3 - 3R_2: [0 \ 0 \ 2 \mid 2 \ -3 \ 1] \Rightarrow R_3/2: [0 \ 0 \ 1 \mid 1 \ -3/2 \ 1/2]$.

$R_2 - 2R_3: [0 \ 1 \ 0 \mid -3 \ 4 \ -1]$. $R_1 \rightarrow R_1 - R_2 - R_3: [1 \ 1 \ 1 \mid 1 \ 0 \ 0] - [0 \ 1 \ 0 \mid -3 \ 4 \ -1] - [0 \ 0 \ 1 \mid 1 \ -3/2 \ 1/2] = [1 \ 0 \ 0 \mid 3 \ -5/2 \ 1/2]$.

Answer: $A^{-1} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix}$.

Problem 11. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

$$\text{tr } A = 6; S_2 = (6 - 2) + (3 + 2) + (2 - 0) = 11; |A| = 1(6 - 2) - 0 + (-1)(2 - 4) = 6. \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0.$$

$\lambda = 1$: rows $(0, 0, -1), (1, 1, 1) \Rightarrow z = 0, x = -y \Rightarrow (1, -1, 0)$.

$\lambda = 2$: rows $(-1, 0, -1), (1, 0, 1), (2, 2, 1) \Rightarrow x = -z, 2x + 2y + z = 0 \Rightarrow$ with $z = -2: x = 2, y = -1 \Rightarrow (2, -1, -2)$.

$\lambda = 3$: rows $(-2, 0, -1), (1, -1, 1) \Rightarrow x : y : z = (0 - 1) : (-1 + 2) : (2 - 0) = -1 : 1 : 2 \Rightarrow (1, -1, -2)$.

Answer: $\lambda = 1: (1, -1, 0); \lambda = 2: (2, -1, -2); \lambda = 3: (1, -1, -2)$.

Problem 12. Verify the Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and find A^{-1} .

$$\lambda^2 - 4\lambda - 5 = 0. A^2 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}. A^2 - 4A - 5I = \begin{bmatrix} 9 - 4 - 5 & 16 - 16 \\ 8 - 8 & 17 - 12 - 5 \end{bmatrix} = O.$$

$$A - 4I - 5A^{-1} = O \Rightarrow A^{-1} = (A - 4I)/5 = (1/5)[-3 \ 4; 2 \ -1].$$

Answer: Verified; $A^{-1} = (1/5)[-3 \ 4; 2 \ -1]$.



Formula Sheet (Last-Minute Revision)

Module I: Basic Calculus

Quantity	Cartesian $y = f(x)$	Parametric / Polar
Arc length	$\int \sqrt{1 + y'^2} dx$	$\int \sqrt{x'^2 + y'^2} dt ; \int \sqrt{r^2 + r'^2} d\theta$
Surface area (x-axis)	$\int 2\pi y \sqrt{1 + y'^2} dx$	$\int 2\pi y \sqrt{x'^2 + y'^2} dt ; \int 2\pi r \sin \theta ds$
Volume (x-axis)	$\int \pi y^2 dx$	$\int \pi y^2 x' dt ; \int (2\pi/3) r^3 \sin \theta d\theta$
Volume (y-axis)	$\int \pi x^2 dy$ or $\int 2\pi xy dx$ (shell)	$\int (2\pi/3) r^3 \cos \theta d\theta$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx; \Gamma(n+1) = n\Gamma(n) = n!; \Gamma(1) = 1; \Gamma(1/2) = \sqrt{\pi}; \Gamma(n)\Gamma(1-n) = \pi/\sin n\pi$$

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \Gamma(m)\Gamma(n)/\Gamma(m+n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta = \int_0^\infty x^{m-1}/(1+x)^{m+n} dx$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta d\theta = \Gamma((p+1)/2)\Gamma((q+1)/2) / [2\Gamma((p+q+2)/2)]$$

$$\int_0^\infty e^{-kx} x^{n-1} dx = \Gamma(n)/k^n; \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$$

$$p\text{-tests: } \int_1^\infty dx/x^p \text{ converges iff } p > 1; \int_0^1 dx/x^p \text{ converges iff } p < 1$$

Standard lengths: astroid $6a$; cardioid $8a$; one arch of cycloid $8a$; circle $2\pi a$.

Module II: Single-variable Calculus

Rolle: $f(a) = f(b) \Rightarrow f'(c) = 0$. LMVT: $f'(c) = [f(b) - f(a)]/(b - a)$. CMVT: $[f(b) - f(a)]/[g(b) - g(a)] = f'(c)/g'(c)$

CMVT special c: $e^x, e^{-x} \rightarrow (a+b)/2$; $\sqrt{x}, 1/\sqrt{x} \rightarrow \sqrt{ab}$; $1/x^2, 1/x \rightarrow 2ab/(a+b)$

Taylor: $f(x) = \sum f^{(n)}(a)(x-a)^n/n!$; Remainder $R_n = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!$

Radius of convergence $R = \lim |c_n/c_{n+1}| = 1/\lim |c_n|^{1/n}$

$e^x = \sum x^n/n!$; $\sin x = x - x^3/6 + x^5/120$; $\cos x = 1 - x^2/2 + x^4/24$; $\tan x = x + x^3/3 + 2x^5/15$

$\log(1+x) = x - x^2/2 + x^3/3 - \dots$; $(1+x)^n = 1 + nx + n(n-1)x^2/2 + \dots$; $\tan^{-1}x = x - x^3/3 + x^5/5$

First derivative test: $+$ to $- \Rightarrow \max$; $-$ to $+$ $\Rightarrow \min$; no change $\Rightarrow$ no extremum.

Module III: Multivariable Calculus

Euler: $x f_x + y f_y = nf$; $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f$

Chain rule: $dz/dt = z_x x' + z_y y'$; implicit: $dy/dx = -f_x/f_y$

$J = \partial(u, v)/\partial(x, y) = u_x v_y - u_y v_x$; $JJ' = 1$; $J = 0 \Rightarrow$ functional dependence

Polar $J = r$; cylindrical $J = r$; spherical $J = r^2 \sin \theta$

$r = f_{xx}, s = f_{xy}, t = f_{yy}$; $rt - s^2 > 0$ & $r < 0 \Rightarrow \text{MAX}$; $rt - s^2 > 0$ & $r > 0 \Rightarrow \text{MIN}$; $rt - s^2 < 0 \Rightarrow \text{SADDLE}$; $= 0 \Rightarrow \text{doubtful}$

Lagrange: $F = f + \lambda\phi$; $F_x = F_y = F_z = 0, \phi = 0$

Module IV: Linear Algebra I

Subspace test: $0 \in W$ and $au + bv \in W$

LI $\Leftrightarrow$ rank of matrix of vectors = number of vectors $\Leftrightarrow$ (n vectors in R^n) determinant $\neq 0$



$\dim R^n = n$; $\dim P_n = n + 1$; $\dim M_{m \times n} = mn$; symmetric $n \times n$: $n(n + 1)/2$; skew: $n(n - 1)/2$

$\text{rank } A = \text{rank } [A|B] = n \Rightarrow \text{unique}$; $= r < n \Rightarrow \text{infinite } (n - r \text{ parameters})$; $\text{rank } A \neq \text{rank } [A|B] \Rightarrow \text{no solution}$

$AX = 0$ has non-trivial solution $\Leftrightarrow |A| = 0$ (square) $\Leftrightarrow \text{rank } A < n$; $\text{rank} + \text{nullity} = n$

Module V: Linear Algebra II

$[A | I] \rightarrow [I | A^{-1}]$ (Gauss-Jordan); $(AB)^{-1} = B^{-1}A^{-1}$

Symmetric $A^T = A$; skew $A^T = -A$; orthogonal $AA^T = I$, $|A| = \pm 1$; $A = (A + A^T)/2 + (A - A^T)/2$

$|A - \lambda I| = 0$; 2×2 : $\lambda^2 - (\text{tr})\lambda + |A| = 0$; 3×3 : $\lambda^3 - (\text{tr})\lambda^2 + S_2\lambda - |A| = 0$

$\Sigma\lambda = \text{trace}$; $\Pi\lambda = |A|$; eigenvalues of A^{-1} : $1/\lambda$; A^m : λ^m ; kA : $k\lambda$; $A + kI$: $\lambda + k$; $\text{adj } A$: $|A|/\lambda$

Cayley-Hamilton: A satisfies its characteristic equation; 2×2 : $A^2 - (\text{tr } A)A + |A|I = O$

Symmetric $\Rightarrow$ real eigenvalues, orthogonal eigenvectors. Skew $\Rightarrow 0$ or purely imaginary.
Orthogonal $\Rightarrow |\lambda| = 1$. Triangular $\Rightarrow$ diagonal entries.